The question was asked by Ronald Brown

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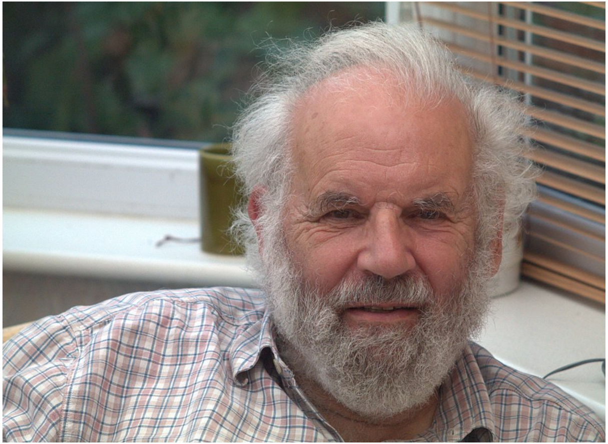

Ronald Brown
1935–2024

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- **What is a double central extension?**
  Here the answer, due to George Janelidze, is:
  a "double extension, central relative to central extensions";
  these appear in the *Hopf formulae* for homology and are classified by cohomology.

What is a double crossed module?

4. Crossed squares [BL87, GWL81, Lod82]

$$\begin{array}{ccc} P & \xrightarrow{p_M} & M \\ {\scriptstyle p_N}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & L \end{array}$$

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X0 $h(mm', n) = {}^{m}h(m', n)h(m, n) \quad$ and $\quad h(m, nn') = h(m, n)^{n}h(m, n')$;

X1 p_M and p_N are L-equivariant, and with the given actions, $(\mu\colon M \to L)$, $(\nu\colon N \to L)$ and $(\mu{\circ}p_M = \nu{\circ}p_N\colon P \to L)$ are crossed modules;

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Morphisms are natural transformations, compatible with the actions and with the map h.
Crossed squares and morphisms between them form the category ***XSqr***.

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Crossed modules are "normalised internal categories in ***Gp***"; indeed, ***XMod*** $\simeq$ ***Cat***$($***Gp***$)$

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The action induces the split extension; M1 iff there is c such that $\mu = c \circ \ker(d)$ and $c \circ e = 1_L$; and M2 is equivalent to the condition that this reflexive graph is an internal category.

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A roundabout answer is that $\boldsymbol{XSqr} \simeq \boldsymbol{Cat}(\boldsymbol{Cat}(\boldsymbol{Gp}))$:
crossed squares are equivalent to
double internal categories (= internal double categories)
via the (de)normalisation procedure applied twice.

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We recall what are semi-abelian categories, and how to define internal actions.

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- any $p \circ i$ where i normal mono (= kernel) and p normal epi can be written as $m \circ e$ with e normal epi and m normal mono;

$$
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- any $p \circ i$ where i normal mono (= kernel) and p normal epi can be written as $m \circ e$ with e normal epi and m normal mono;
- whenever $M \xrightarrow{k} X \underset{s}{\overset{d}{\rightleftarrows}} L$ where $k = \ker(d)$ and $d \circ s = 1_L$, k and s are jointly extremal-epic. Hence $d = \operatorname{coker}(k)$.

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Semi-abelian categories may be described in terms of "good behaviour" of their kernels, cokernels and split **epi**morphisms.

For this, we need a **zero object**: an object 0 which is initial and terminal. We further assume:

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Examples:

- abelian categories: modules over a ring, sheaves of abelian groups;
- pointed varieties of universal algebras with a group operation: groups, rings, Lie algebras, associative algebras, crossed modules;
- loops, Heyting semilattices, cocommutative Hopf algebras, Set_*^{op}.

8. Cosmash products, commutators and actions [MM10, CJ03, HVdL13]

In a semi-abelian category $\mathscr{X}$, any two objects X and Y induce a short exact sequence

$$0 \longrightarrow X \diamond Y \xrightarrow{h_{X,Y}} X + Y \xrightarrow{\Sigma_{X,Y}} X \times Y \longrightarrow 0$$

where $X \diamond Y$ is called the **cosmash product** of X and Y. It is a measure of non-abelianness.

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As in $\boldsymbol{Gp}$, split extensions correspond to **action cores** via semi-direct products: $X \cong M \rtimes_{\psi} L$.

In $\boldsymbol{Gp}$, $X \diamond Y$ is the subgroup of $X + Y$ generated by formal commutator elements $xyx^{-1}y^{-1}$. The morphism ψ sends $\ell m \ell^{-1} m^{-1}$ to ${}^{\ell}m m^{-1}$.

6. In which sense is a crossed square a "double crossed module"?

A roundabout answer is that $\boldsymbol{XSqr} \simeq \boldsymbol{Cat}(\boldsymbol{Cat}(\boldsymbol{Gp}))$:
crossed squares are equivalent to
double internal categories (= internal double categories)
via the (de)normalisation procedure applied twice.

$$
\begin{tikzcd}
Z \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d, shift right=2] & M \rtimes L \arrow[l] \arrow[d, shift right=2, "c_M"'] \arrow[d, shift left=2, "d_M"] \\
N \rtimes L \arrow[u] \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] & L \arrow[l, "e_N" description] \arrow[u, "e_M" description]
\end{tikzcd}
$$

In order for the more direct $\boldsymbol{XSqr} \simeq \boldsymbol{XMod}(\boldsymbol{XMod}(\boldsymbol{Gp}))$ to make sense,
we need to understand what is an **internal crossed module**.

We define $\boldsymbol{XMod}(\mathscr{X}) \simeq \boldsymbol{Cat}(\mathscr{X})$ where $\mathscr{X}$ is a semi-abelian category;
we see that $\boldsymbol{XMod} \simeq \boldsymbol{XMod}(\boldsymbol{Gp})$.

Since $\boldsymbol{XMod}(\mathscr{X})$ is again semi-abelian,
we may put $\boldsymbol{XSqr}(\mathscr{X}) \coloneqq \boldsymbol{XMod}(\boldsymbol{XMod}(\mathscr{X}))$ and obtain $\boldsymbol{XSqr} \simeq \boldsymbol{XSqr}(\boldsymbol{Gp})$.

5. Crossed modules [Whi41]

A **crossed module (of groups)** is a morphism $\mu\colon M \to L$ with an action of L on M such that for all $\ell \in L$ and $m, m' \in M$:

M1 $\mu({}^{\ell}m) = {}^{\ell}\mu(m)$

M2 ${}^{\mu(m)}m' = {}^{m}m'$

Morphisms are equivariant natural transformations. This defines the category ***XMod***.

Special cases

- μ injective: it is a normal subgroup inclusion, with the conjugation action ${}^{\ell}m = \ell m \ell^{-1}$;
- μ surjective: it is a central extension, so
$$0 = [\mathrm{Ker}(\mu), M] = \langle kmk^{-1}m^{-1} \mid k, m \in M, \mu(k) = 1 \rangle;$$
we may put ${}^{\ell}m' = mm'm^{-1}$ for any $m \in M$ such that $\mu(m) = \ell$.

Crossed modules are "normalised internal categories in ***Gp***"; indeed, ***XMod*** $\simeq$ ***Cat***$($***Gp***$)$

$$\begin{tikzcd} M \arrow[r, tail, "\ker(d)"] & M \rtimes L \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, dotted, "c"'] & L \arrow[l, "e" description] \end{tikzcd}$$

The action induces the split extension; M1 iff there is c such that $\mu = c \circ \ker(d)$ and $c \circ e = 1_L$; and M2 is equivalent to the condition that this reflexive graph is an internal category.

9. Internal crossed modules [Jan03, HVdL13]

The aim is to have an equivalence $\boldsymbol{XMod}(\mathscr{X}) \simeq \boldsymbol{Cat}(\mathscr{X})$ for any semi-abelian category $\mathscr{X}$.

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10. The ternary cosmash product [HVdL13, Hig56, CJ03, CGVdL15]

In a semi-abelian category $\mathscr{X}$, any three objects X, Y and Z give rise to a morphism

$$\begin{pmatrix} \iota_X & \iota_Y & 0 \\ \iota_X & 0 & \iota_Z \\ 0 & \iota_Y & \iota_Z \end{pmatrix} : X + Y + Z \longrightarrow (X+Y) \times (X+Z) \times (Y+Z)$$

and its kernel $h_{X,Y,Z} \colon X \diamond Y \diamond Z \to X + Y + Z$.

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The codiagonal induces folding maps $S^{L,M}_{1,2} \colon L \diamond M \diamond M \to L \diamond M$ and $S^{L,M}_{2,1} \colon L \diamond L \diamond M \to L \diamond M$.

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$\psi_{1,2} = \psi \circ S^{L,M}_{1,2} \qquad \psi_{2,1} = \psi \circ S^{L,M}_{2,1}$

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Unfortunately, this doesn't explain the Brown–Loday definition at all!

4. Crossed squares [BL87, GWL81, Lod82]

$$\begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

A **crossed square** is a commuting square in the category ***Gp*** of groups, with

- actions of L on M, N and P
  (of M on P and N via μ, of N on M and P via ν)
- and a function $h\colon M \times N \to P$

such that for all $\ell \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$:

X0 $h(mm', n) = {}^{m}h(m', n)h(m, n) \quad$ and $\quad h(m, nn') = h(m, n)^{n}h(m, n')$;

X1 p_M and p_N are L-equivariant, and with the given actions, $(\mu\colon M \to L)$, $(\nu\colon N \to L)$ and $(\mu{\circ}p_M = \nu{\circ}p_N\colon P \to L)$ are crossed modules;

X2 $p_M(h(m, n)) = m^{n}m^{-1} \quad$ and $\quad p_N(h(m, n)) = {}^{m}nn^{-1}$;

X3 $h(p_M(p), n) = p^{n}p^{-1} \quad$ and $\quad h(m, p_N(p)) = {}^{m}pp^{-1}$;

X4 ${}^{\ell}h(m, n) = h({}^{\ell}m, {}^{\ell}n)$.

Morphisms are natural transformations, compatible with the actions and with the map h.
Crossed squares and morphisms between them form the category ***XSqr***.

11. Internal crossed squares [dMVdL20]

By definition now, $\boldsymbol{XSqr}(\mathscr{X}) := \boldsymbol{XMod}(\boldsymbol{XMod}(\mathscr{X}))$ for any semi-abelian category $\mathscr{X}$; then $\boldsymbol{XSqr} \simeq \boldsymbol{XSqr}(\boldsymbol{Gp})$ is automatic.

Unfortunately, this doesn't explain the Brown–Loday definition at all!

Our attempt at a more detailed analysis depends on the **non-abelian tensor product**, also introduced by Brown and Loday in the article [BL87].

12. The non-abelian tensor product of groups [BL87]

Given two groups M and N acting on each other (and on themselves by conjugation), their **non-abelian tensor product** $M \otimes N$ is the group generated by the symbols $m \otimes n$ for $m \in M$ and $n \in N$, subject to the relations

$$(mm') \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n) \qquad m \otimes (nn') = (m \otimes n)({}^n m \otimes {}^n n')$$

for all $m, m' \in M$ and $n, n' \in N$.

13. The tensor product at work [BL87]

$$\begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

We consider the crossed square on the left; in particular, we have

- crossed modules μ and ν, and
- a function $h\colon M \times N \to P$, where

X0 $h(mm', n) = {}^m h(m', n) h(m, n) \quad \text{and} \quad h(m, nn') = h(m, n)\, {}^n h(m, n')$;

X4 ${}^\ell h(m, n) = h({}^\ell m, {}^\ell n)$.

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X4 ${}^{\ell}h(m, n) = h({}^{\ell}m, {}^{\ell}n).$

The induced $M \otimes N$ is the group generated by $m \otimes n$ for $m \in M$ and $n \in N$, such that

$$(mm') \otimes n = ({}^{m}m' \otimes {}^{m}n)(m \otimes n) \qquad \text{and} \qquad m \otimes (nn') = (m \otimes n)({}^{n}m \otimes {}^{n}n').$$

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The function h induces a morphism $\bar{h}\colon M \otimes N \to P\colon m \otimes n \mapsto h(m, n)$, because

$$\begin{aligned} \bar{h}((mm') \otimes n) &= h(mm', n) = {}^m h(m', n)h(m, n) = {}^{\mu(m)} h(m', n)h(m, n) \\ &= h({}^{\mu(m)} m', {}^{\mu(m)} n)h(m, n) = h({}^m m', {}^m n)h(m, n) \\ &= \bar{h}({}^m m' \otimes {}^m n)\bar{h}(m \otimes n) = \bar{h}(({}^m m' \otimes {}^m n)(m \otimes n)) \end{aligned}$$

and, likewise, $\bar{h}(m \otimes (nn')) = \bar{h}((m \otimes n)({}^n m \otimes {}^n n'))$.

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It is common to restrict ourselves to the following key special case, which in the terminology of Brown–Loday amounts to asking that the given actions are *compatible*:

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It is common to restrict ourselves to the following key special case, which in the terminology of Brown–Loday amounts to asking that the given actions are *compatible*:

We have a group L and two L-crossed modules $\mu\colon M \to L$ and $\nu\colon N \to L$; these induce actions of M and N on each other, and we obtain a crossed module $M \otimes N \to L$. Thus, the non-abelian tensor product restricts to a functor

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How to extend this beyond the case of groups?

14. Characterising $\otimes$ via a universal property I [BL87]

Let $\mu\colon M \to L$ and $\nu\colon N \to L$ be L-crossed modules of groups.
Then the crossed square on the left

$$\begin{tikzcd} M \otimes N \arrow[r, "\pi_M"] \arrow[d, "\pi_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd} \xrightarrow{\left(\begin{smallmatrix} \phi & 1_M \\ 1_N & 1_L \end{smallmatrix}\right)} \begin{tikzcd} P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}$$

where $\pi_M(m \otimes n) = m^n m^{-1}$, $\pi_N(m \otimes n) = {}^m n n^{-1}$ and $h(m, n) = m \otimes n$
is universal in the following sense:

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If the square on the right is another crossed square (with the same μ and ν), then there is a unique morphism of crossed squares $\left(\begin{smallmatrix} \phi & 1_M \\ 1_N & 1_L \end{smallmatrix}\right)$ from the left-hand to the right-hand crossed square which is the identity on M, N and L and where $\phi\colon M \otimes N \to P$.

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This allows us to charaterise $\otimes$ as a pushout in *XSqr*.

15. Characterising $\otimes$ via a universal property II [BL87]

Let $\mu\colon M \to L$ and $\nu\colon N \to L$ be L-crossed modules of groups.

Then the diagram

$$\begin{array}{ccc}
\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & L \end{tikzcd}
& \xrightarrow{\left(\begin{smallmatrix} 1_0 & 0 \\ 1_0 & 1_L \end{smallmatrix}\right)} &
\begin{tikzcd} 0 \arrow[r] \arrow[d] & M \arrow[d, "\mu"] \\ 0 \arrow[r] & L \end{tikzcd}
\\
\left(\begin{smallmatrix} 1_0 & 1_0 \\ 0 & 1_L \end{smallmatrix}\right) \Big\downarrow & & \Big\downarrow \left(\begin{smallmatrix} 0 & 1_M \\ 0 & 1_L \end{smallmatrix}\right)
\\
\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}
& \xrightarrow[\left(\begin{smallmatrix} 0 & 0 \\ 1_N & 1_L \end{smallmatrix}\right)]{} &
\begin{tikzcd} M \otimes N \arrow[r, "\pi_M"] \arrow[d, "\pi_N"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & L \end{tikzcd}
\end{array}$$

a pushout in ***XSqr***.

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Let $\mu\colon M \to L$ and $\nu\colon N \to L$ be L-crossed modules of groups.

Then the diagram

$$
\begin{tikzcd}
\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & L \end{array}
\arrow[r, "{\begin{pmatrix} 1_0 & 0 \\ 1_0 & 1_L \end{pmatrix}}"]
\arrow[d, "{\begin{pmatrix} 1_0 & 1_0 \\ 0 & 1_L \end{pmatrix}}"']
&
\begin{array}{ccc} 0 & \longrightarrow & M \\ \big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ 0 & \longrightarrow & L \end{array}
\arrow[d, "{\begin{pmatrix} 0 & 1_M \\ 0 & 1_L \end{pmatrix}}"]
\\
\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow \\ N & \xrightarrow[\nu]{} & L \end{array}
\arrow[r, "{\begin{pmatrix} 0 & 0 \\ 1_N & 1_L \end{pmatrix}}"']
&
\begin{array}{ccc} M \otimes N & \xrightarrow{\pi_M} & M \\ {\scriptstyle \pi_N}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & L \end{array}
\end{tikzcd}
$$

a pushout in ***XSqr***.

This, we can do in general!

16. Characterising $\otimes$ via a universal property III [dMVdL20]

Let $\mu\colon M \to L$ and $\nu\colon N \to L$ be L-crossed modules in a semi-abelian category $\mathscr{X}$.
Consider their induced internal category structures

$$\begin{tikzcd} N \arrow[r, triangle, "k_N"] & N \rtimes L \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] & L \arrow[l, "e_N" description] \arrow[r, "e_M" description] & M \rtimes L \arrow[l, shift right=2, "c_M"'] \arrow[l, shift left=2, "d_M"] & M. \arrow[l, triangle, "k_M"'] \end{tikzcd}$$

Let $\mu\colon M \to L$ and $\nu\colon N \to L$ be L-crossed modules in a semi-abelian category $\mathscr{X}$.
Consider their induced internal category structures

$$
\begin{tikzcd}
N \arrow[r, tail, "k_N"] & N \rtimes L \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] & L \arrow[l, "e_N"] \arrow[r, "e_M"] & M \rtimes L \arrow[l, shift right=2, "c_M"'] \arrow[l, shift left=2, "d_M"] & M. \arrow[l, tail, "k_M"']
\end{tikzcd}
$$

In $\boldsymbol{Cat}^2(\mathscr{X})$, we construct the following span.

$$
\begin{tikzcd}
N \rtimes L \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] \arrow[dd, shift right=2, "1"'] \arrow[dd, shift left=2, "1"] & L \arrow[l, "e_N"] \arrow[dd, shift right=2, "1"'] \arrow[dd, shift left=2, "1"] & & L \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] \arrow[dd, shift right=2, "1"'] \arrow[dd, shift left=2, "1"] & L \arrow[l, "1"] \arrow[dd, shift right=2, "1"'] \arrow[dd, shift left=2, "1"] & & M \rtimes L \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] \arrow[dd, shift right=2, "c_M"'] \arrow[dd, shift left=2, "d_M"] & M \rtimes L \arrow[l, "1"] \arrow[dd, shift right=2, "c_M"'] \arrow[dd, shift left=2, "d_M"] \\
& & {} & & & {} & & \\
N \rtimes L \arrow[r, shift left=2, "d_N"] \arrow[r, shift right=2, "c_N"'] \arrow[uu, "1"] & L \arrow[l, "e_N"] \arrow[uu, "1"] & & L \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] \arrow[uu, "1"] & L \arrow[l, "1"] \arrow[uu, "1"] & & L \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] \arrow[uu, "e_M"] & L \arrow[l, "1"] \arrow[uu, "e_M"]
\arrow[from=2-3, to=2-2, "\left(\begin{smallmatrix} e_N & 1_L \\ e_N & 1_L \end{smallmatrix}\right)"']
\arrow[from=2-6, to=2-7, "\left(\begin{smallmatrix} e_M & e_M \\ 1_L & 1_L \end{smallmatrix}\right)"]
\end{tikzcd}
$$

16. Characterising $\otimes$ via a universal property III [dMVdL20]

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& & {} \arrow[l, "\begin{pmatrix} e_N & 1_L \\ e_N & 1_L \end{pmatrix}"'] & & & {} \arrow[r, "\begin{pmatrix} e_M & e_M \\ 1_L & 1_L \end{pmatrix}"] & & \\
N \rtimes L \arrow[r, shift left=3, "d_N"] \arrow[r, shift right=3, "c_N"'] \arrow[uu, "1" description] & L \arrow[l, "e_N" description] \arrow[uu, "1" description] & & L \arrow[r, shift left=3, "1"] \arrow[r, shift right=3, "1"'] \arrow[uu, "1" description] & L \arrow[l, "1" description] \arrow[uu, "1" description] & & L \arrow[r, shift left=3, "1"] \arrow[r, shift right=3, "1"'] \arrow[uu, "e_M" description] & L \arrow[l, "1" description] \arrow[uu, "e_M" description]
\end{tikzcd}$$

The needed tensor crossed square in $\mathscr{X}$ is the normalisation of the span's pushout.

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This defines a functor $\otimes\colon \boldsymbol{XMod}_L(\mathscr{X}) \times \boldsymbol{XMod}_L(\mathscr{X}) \to \boldsymbol{XMod}_L(\mathscr{X})$.

17. Some examples [dMVdL20, BL87, Mac60]

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- In an algebraically coherent semi-abelian category,

$$\begin{tikzcd} {[M,N]} \arrow[r, tail] \arrow[d, tail] & N \arrow[d, tail, "n"] \\ M \arrow[r, tail, "m"'] & L \end{tikzcd}$$

is the initial crossed square *of normal monomorphisms* over m and n.

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The intersection $M \wedge N$ also forms such a crossed square, and we may show that the image of the canonical map $\bar{h}\colon M \otimes N \to M \wedge N$ is $[M, N]$.

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18. Internal crossed squares II [dMVdL20]

Let $\mathscr{X}$ be an algebraically coherent semi-abelian category.
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In all examples we know of, the converse holds.
We don't know if this is true in general.

What is a double central extension?

20. Central extensions and the Hopf formula for H_2 [Hop42, EVdL04]

An **extension** under K and over L is a short exact sequence

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How to extend this to $H_n(L)$ for $n \geqslant 3$?

21. The Hopf formula for H_3 [BE88, DIP05, EGVdL08]

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d, tail] & K_N \arrow[r, two heads] \arrow[d, tail] & K_\mu \arrow[d, tail] \\ K_M \arrow[r, tail] \arrow[d, two heads] & P \arrow[r, two heads, "p_M"] \arrow[d, two heads, "p_N"'] & M \arrow[d, two heads, "\mu"] \\ K_\nu \arrow[r, tail] & N \arrow[r, two heads, "\nu"'] & L \end{tikzcd}$$

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so that its kernel $K_M \vee K_N$ commutes with P:
this condition is much stronger than what we need!
(Note $[K, P] \vee [K_M, K_N] \leqslant [K_M \vee K_N, P] = 0$.)

“Central extension of central extensions”, so $[K_M, P] = 0 = [K_N, P]$, agrees with this.

An example of a square which should induce a double central extension is the one on the right, where M and N are arbitrary groups.
Indeed, $[0 \times N, M \times 0] = 0 = [(M \times 0) \wedge (0 \times N), M \times N]$.
It is not a crossed square though, unless M and N are abelian.

$$
\begin{tikzcd}
M \times N \arrow[r, "\pi_M"] \arrow[d, "\pi_N"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\nu"'] & 0
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22. Some attempts towards an answer

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d, tail] & K_N \arrow[r] \arrow[d, tail] & K_\mu \arrow[d, tail] \\ K_M \arrow[r, tail] \arrow[d, tail] & P \arrow[r, "p_M"] \arrow[d, "p_N"'] & M \arrow[d, "\mu"] \\ K_\nu \arrow[r, tail] & N \arrow[r, "\nu"'] & L \end{tikzcd}$$

We need that $[K, P] \vee [K_M, K_N] = 0$,
so both $[K, P]$ and $[K_M, K_N]$ must be trivial.

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What is a double central extension?

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$$\begin{tikzcd}
Eq(\mu) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "\eta_{Eq(\mu)}"'] & M \arrow[l] \arrow[r, "\mu", -{Triangle[open]}] \arrow[d, "\eta_M"] & L \\
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This idea enables an algebraic proof of the Hopf formulae in all dimensions.

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Given a double extension in $\mathscr{X}$,

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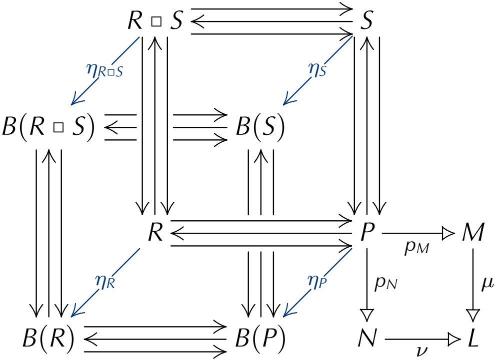

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This idea works in all semi-abelian categories, for extensions of arbitrary dimension.

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  Without his contributions, my work would be incomparably less exciting.
  For this, I will forever remain immensely grateful.

Thank you!

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