Universitat de València

FREE MONOIDS AND RIGUET CONGRUENCES

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University of Aveiro
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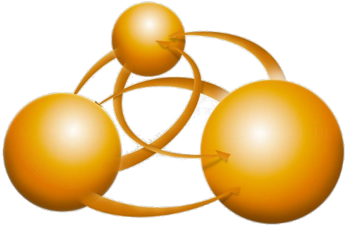

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PRELIMINARIES

Many-sorted sets and the free monoid on a set

MANY-SORTED SETS

Given a set A, an A**-sorted set** is an A-indexed family $X = (X_a)_{a\in A}$ where, for every $a \in A$, X_a is a set.

If X and Y are A-sorted sets, an A**-sorted mapping** from X to Y is an A-indexed family $f = (f_a)_{a\in A}$ where, for every $a \in A$, f_a is a mapping from X_a to Y_a.

A-sorted sets and A-sorted mappings form a category Set^A.

An A-sorted set X is **finite** if $\coprod X$ is finite.

Finite A-sorted sets and A-sorted mappings between them form a category $\mathsf{Set}_{\mathrm{f}}^A$.

FREE MONOID ON A SET

Given a set A, a **word** on A is a mapping $\mathbf{a}\colon n \to A$ for some $n \in \mathbb{N}$. We will denote by $A^\star$ the set of all words on A. $|\mathbf{a}|$ is the domain of the word $\mathbf{a}$. Moreover, for a letter $a \in A$ we will denote by $|\mathbf{a}|_a$ the number of ocurrences of a in $\mathbf{a}$.

Two words $\mathbf{a}$ and $\mathbf{b}$ on A can be concatenated $\mathbf{a} \curlywedge \mathbf{b}$.

The empty word will be denoted by λ.

The **free monoid on** A is $(A^\star, \curlywedge, \lambda)$.

Example

$\mathbf{a} = baac$ and $\mathbf{b} = caba$ are words on $A = \{a, b, c\}$.

FREE MONOID ON A SET

Proposition

$$\begin{CD} A @>{\eta_A}>> A^\star \\ @V{f}VV @VV{\exists ! f^{@}}V \\ B @>>{\eta_B}> B^\star \end{CD}$$

FREE MONOID ON A SET

Example

If we consider the sets $A = \{a, b, c\}$ and $B = \{x, y, z\}$ and the mapping $f\colon A \to B$ defined as follows, $f(a) = x$, $f(b) = y$ and $f(c) = z$, then:

$$\begin{aligned} f^{@}(\mathbf{a}) &= f^{@}(baac) \\ &= f(b)f(a)f(a)f(c) \\ &= yxxz \end{aligned}$$

FREE MONOID ON A SET

We will say that two words $\mathbf{a}$ and $\mathbf{b}$ are **congruent**, $\mathbf{a} \equiv^A \mathbf{b}$, if, for every $a \in A$, $|\mathbf{a}|_a = |\mathbf{b}|_a$.

Example

The words $\mathbf{a} = baac$ and $\mathbf{b} = caba$ are congruent.

FREE MONOID ON A SET

For a word $\mathbf{a}$ and an index $i \in |\mathbf{a}|$, we will denote by $\mathrm{occ}_{\mathbf{a}}(i)$ the number of occurrence of $\mathbf{a}(i)$ in $\mathbf{a}$.

For a word $\mathbf{a}$, a letter a occurring in $\mathbf{a}$ and $j \in |\mathbf{a}|_a$, we will denote by $\mathrm{pos}_{\mathbf{a},a}(j)$ the position of the j-th occurrence of a in $\mathbf{a}$.

Example

For the word $\mathbf{a} = baac$, $\mathrm{occ}_{\mathbf{a}}(2) = 1$ and $\mathrm{pos}_{\mathbf{a},c}(0) = 3$.

FREE MONOID ON A SET

For every pair of congruent words $\mathbf{a} \equiv^A \mathbf{b}$, the permutation

$$\begin{array}{rccc} \sigma_{\mathbf{a},\mathbf{b}}: & |\mathbf{a}| & \to & |\mathbf{b}| \\ & i & \mapsto & \mathsf{pos}_{\mathbf{b},\mathbf{a}(i)}(\mathsf{occ}_{\mathbf{a}}(i)) \end{array}$$

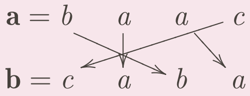
is called the **canonical permutation** for the pair $(\mathbf{a}, \mathbf{b})$

Proposition

The canonical permutation is such that $\mathbf{a} = \mathbf{b} \circ \sigma_{\mathbf{a},\mathbf{b}}$.

FREE MONOID ON A SET

Example



Proposition

If $\mathbf{a} \equiv^A \mathbf{b}$ and $\mathbf{b} \equiv^A \mathbf{c}$, then

$$(1)\sigma_{\mathbf{a},\mathbf{a}} = \mathrm{id}_{|\mathbf{a}|} \qquad (2)\sigma_{\mathbf{b},\mathbf{c}} \circ \sigma_{\mathbf{a},\mathbf{b}} = \sigma_{\mathbf{a},\mathbf{c}} \qquad (3)\sigma^{-1}_{\mathbf{a},\mathbf{b}} = \sigma_{\mathbf{b},\mathbf{a}}$$

THE CATEGORY $\mathsf{C}(\mathbf{A}^\star)$

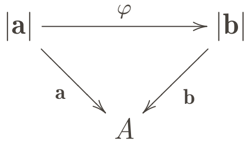
The first category equivalent to $\mathsf{Set}_{\mathrm{f}}^{A}$

THE CATEGORY C($\mathbf{A}^\star$)

Let A be a set. The category C($\mathbf{A}^\star$) is defined as follows.

Objects. Words on A.

Morphisms. A morphism from $\mathbf{a}$ to $\mathbf{b}$ is a mapping $\varphi\colon |\mathbf{a}| \to |\mathbf{b}|$ such that the diagram



commutes.

If $|A| \geq 2$, the category C($\mathbf{A}^\star$) is not skeletal.

THE FUNCTOR $\mathbb{C}$ FROM Set TO Cat

$$\mathsf{Set} \xrightarrow{\ \mathrm{C}\ } \mathsf{Cat} \qquad\qquad \mathsf{C}(\mathbf{A}^\star) \xrightarrow{\ \mathrm{C}(f)\ } \mathsf{C}(\mathbf{B}^\star)$$

$$\begin{array}{ccc} A & & \mathsf{C}(\mathbf{A}^\star) \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \mathrm{C}(f)} \\ B & & \mathsf{C}(\mathbf{B}^\star) \end{array} \quad \text{where} \quad \begin{array}{ccc} \mathbf{a} & & f^{@}(\mathbf{a}) \\ {\scriptstyle \varphi}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \varphi} \\ \mathbf{b} & & f^{@}(\mathbf{b}) \end{array}$$

$\mathsf{C}(\mathbf{A}^\star) \simeq \mathsf{Set}_{\mathrm{f}}^{A}$

Proposition

The categories $\mathsf{C}(\mathbf{A}^\star)$ and $\mathsf{Set}_{\mathrm{f}}^{A}$ are equivalent.

The equivalence is

$$\mathsf{C}(\mathbf{A}^\star) \xrightarrow{\ \downarrow^{A}(\bullet)\ } \mathsf{Set}_{\mathrm{f}}^{A}$$

$$\begin{array}{ccc} \mathbf{a} & & (\mathbf{a}^{-1}[\{a\}])_{a\in A} \\ \varphi \downarrow & \longmapsto & \downarrow (\varphi)_{a\in A} \\ \mathbf{b} & & (\mathbf{b}^{-1}[\{a\}])_{a\in A} \end{array}$$

THE CATEGORY $\mathrm{Q}(\mathbf{A}^{\star})$

A Riguet quotient of $\mathrm{C}(\mathbf{A}^{\star})$

RIGUET CONGRUENCE

Riguet 1960 [2]

A Riguet congruence on a category C is an ordered pair

$$\Phi = \left(\Phi^{\mathrm{ob}}, \left(\Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)}\right)_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right) \in \Phi^{\mathrm{ob}} \times \Phi^{\mathrm{ob}}}\right)$$

in which

- Φ^{ob} is an equivalence relation on $\mathrm{Ob}(\mathsf{C})$ and,
- for every matrix $\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right) \in \Phi^{\mathrm{ob}} \times \Phi^{\mathrm{ob}}$, in which we agree that $(a, a') \in \Phi^{\mathrm{ob}}$ and $(b, b') \in \Phi^{\mathrm{ob}}$, $\Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)}$ is a subset of $\mathrm{Hom}_{\mathsf{C}}(a, b) \times \mathrm{Hom}_{\mathsf{C}}(a', b')$ such that

RIGUET CONGRUENCE

for every $(a,a'),(a',a''),(b,b'),(b',b''),(c,c') \in \Phi^{\mathrm{ob}}$

1. $\Delta_{\mathrm{Hom}_{\mathsf{C}}(a,b)} \subseteq \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right)}$;
2. $(\Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)})^{-1} = \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a' & b' \\ a & b \end{smallmatrix}\right)}$;
3. $\Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a' & b' \\ a'' & b'' \end{smallmatrix}\right)} \circ \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)} \subseteq \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a'' & b'' \end{smallmatrix}\right)}$;
4. if $(f,f') \in \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)}$ and $(g,g') \in \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} b & c \\ b' & c' \end{smallmatrix}\right)}$ then $(g \circ f, g' \circ f') \in \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & c \\ a' & c' \end{smallmatrix}\right)}$;
5. if $f\colon a \to b$, then there exists an $f'\colon a' \to b'$ such that $(f,f') \in \Phi^{\mathrm{fl}}_{\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)}$.

RIGUET QUOTIENT

For a Riguet congruence Φ on a category C we define its quotient relative to Φ is defined as follows

Objects. $\mathrm{Ob}(\mathsf{C}/\Phi) = \mathrm{Ob}(\mathsf{C})/\Phi^{\mathrm{ob}}$

Morphisms. $\mathrm{Hom}_{\mathsf{C}/\Phi}([a],[b]) = \{[f] \mid f \in \mathrm{Hom}_{\mathsf{C}}(a,b)\}$ where

$$[f] = \bigcup_{a'\in[a],b'\in[b]} \left\{ f' : a' \to b' \mid (f, f') \in \Phi^{\mathrm{fl}}_{\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}} \right\}$$

Proposition

The projection functor pr_Φ is a functor from C to C/Φ.

RIGUET CONGRUENCE ON C($\mathbf{A}^\star$)

We consider the following binary relations on objects and morphisms of C($\mathbf{A}^\star$):

Objects. $\mathbf{a} \equiv^A \mathbf{b}$.

Morphisms. If $\mathbf{a} \equiv^A \mathbf{a}'$ and $\mathbf{b} \equiv^A \mathbf{b}'$, then $\varphi\colon \mathbf{a} \to \mathbf{b}$ and $\varphi'\colon \mathbf{a}' \to \mathbf{b}'$ are such that $\varphi \equiv^A_{\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{a}' & \mathbf{b}'\end{smallmatrix}\right)} \varphi'$ if the following diagram commutes

$$\begin{array}{ccc} |\mathbf{a}| & \xrightarrow{\ \varphi\ } & |\mathbf{b}| \\ {\scriptstyle \sigma_{\mathbf{a},\mathbf{a}'}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{\mathbf{b},\mathbf{b}'}} \\ |\mathbf{a}'| & \xrightarrow[\ \varphi'\ ]{} & |\mathbf{b}'| \end{array}$$

THE CATEGORY Q$(\mathbf{A}^\star)$

Proposition

$$\left(\equiv^A, \left(\equiv^A_{\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{a}' & \mathbf{b}'\end{smallmatrix}\right)}\right)_{\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{a}' & \mathbf{b}'\end{smallmatrix}\right)\in\equiv^A\times\equiv^A}\right)$$ is a Riguet congruence on C$(\mathbf{A}^\star)$.

We let Q$(\mathbf{A}^\star)$ stand for the quotient category C$(\mathbf{A}^\star)/\equiv^A$.

The category Q$(\mathbf{A}^\star)$ is skeletal.

THE FUNCTOR Q FROM Set TO Cat

$$\mathsf{Set} \xrightarrow{\;Q\;} \mathsf{Cat} \qquad\qquad \mathrm{Q}(\mathbf{A}^\star) \xrightarrow{\;Q(f)\;} \mathrm{Q}(\mathbf{B}^\star)$$

$$\begin{array}{ccc} A & & \mathrm{Q}(\mathbf{A}^\star) \\ {\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle Q(f)} \\ B & & \mathrm{Q}(\mathbf{B}^\star) \end{array} \qquad \text{where} \qquad \begin{array}{ccc} [\mathbf{a}] & & [f^{@}(\mathbf{a})] \\ {\scriptstyle [\varphi]}\downarrow & \longmapsto & \downarrow{\scriptstyle [\varphi]} \\ [\mathbf{b}] & & [f^{@}(\mathbf{b})] \end{array}$$

THE MAPPING $|\varphi|_a$

For every mapping $\varphi\colon \mathbf{a} \to \mathbf{b}$ in C($\mathbf{A}^\star$) and every a in A,

$$\begin{array}{ccc} |\mathbf{a}| & \xrightarrow{\ \varphi\ } & |\mathbf{b}| \\ {\scriptstyle \mathrm{pos}_{\mathbf{a},a}}\downarrow & & \downarrow{\scriptstyle \mathrm{pos}_{\mathbf{b},a}} \\ |\mathbf{a}|_a & \underset{\exists! |\varphi|_a}{\dashrightarrow} & |\mathbf{b}|_a \end{array}$$

Proposition

If $\varphi \equiv^{A}_{\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b} \\ \mathbf{a}' & \mathbf{b}'\end{smallmatrix}\right)} \varphi'$, then, for every a in A, $|\varphi|_a = |\varphi'|_a$.

Q($\mathbf{A}^\star$) $\simeq$ Set$_\mathrm{f}^A$

Proposition

The categories Q($\mathbf{A}^\star$) and Set$_\mathrm{f}^A$ are equivalent.

The equivalence is

$$\mathsf{Q}(\mathbf{A}^\star) \xrightarrow{(|\bullet|_a)_{a\in A}} \mathsf{Set}_\mathrm{f}^A$$

$$\begin{array}{ccc} [\mathbf{a}] & & (|\mathbf{a}|_a)_{a\in A} \\ {\scriptstyle [\varphi]}\downarrow & \longmapsto & \downarrow{\scriptstyle (|\varphi|_a)_{a\in A}} \\ [\mathbf{b}] & & (|\mathbf{b}|_a)_{a\in A} \end{array}$$

CONCLUSIONS

1. $\mathsf{Set}_{\mathrm{f}}^{A} \simeq \mathsf{C}(\mathbf{A}^{\star}) \simeq \mathsf{Q}(\mathbf{A}^{\star})$.
2. $\mathsf{Q}(\mathbf{A}^{\star})$ plays an analogous role for the category $\mathsf{Set}_{\mathrm{f}}^{A}$ as $\mathsf{Card}_{\mathrm{f}}$ does for $\mathsf{Set}_{\mathrm{f}}$.
3. To the best of our knowledge, it is the first example of a non-trivial Riguet congruence.

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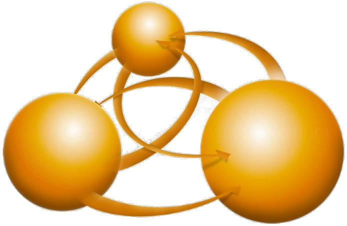

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