A graphical calculus for linear categories

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Plan of the talk

Linear categories

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Linear-nonlinear adjunctions

$$\mathcal{C} = (\mathcal{C}, \times, 1, \Rightarrow) \quad \perp \quad \mathcal{L} = (\mathcal{L}, \otimes, \top, \multimap)$$

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Of our particular interest are **linear categories**

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Problem (technical nightmare of linear categories)

There has been no graphical calculi for linear categories for 38 years.

Plan of the talk

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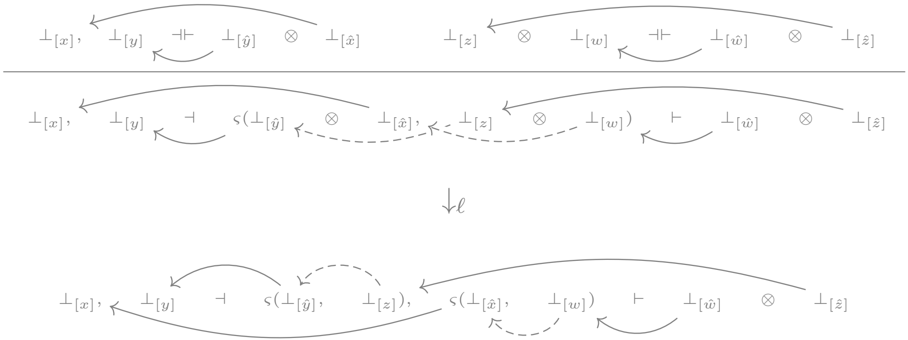

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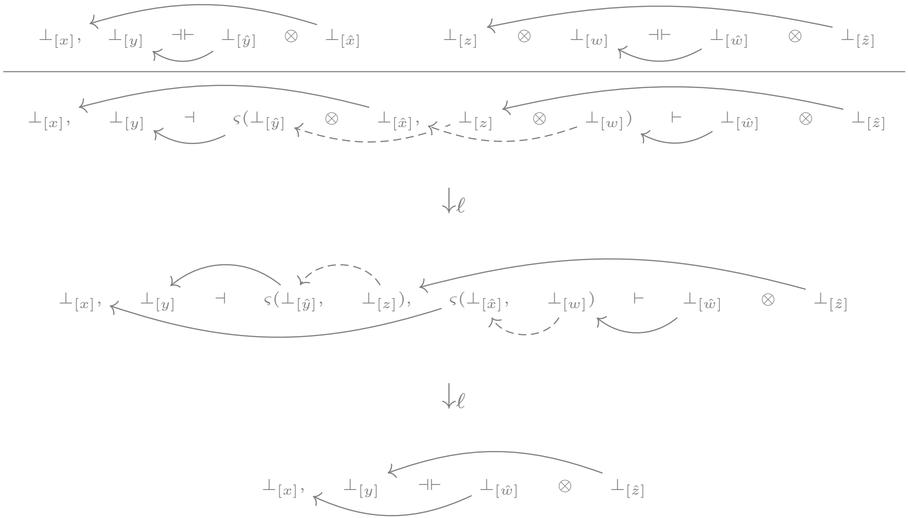
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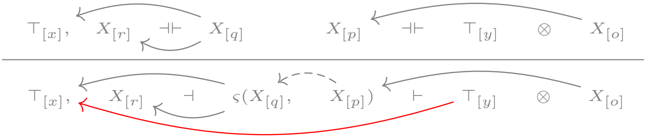

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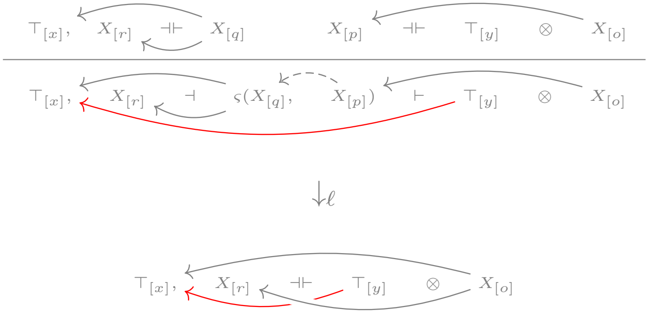
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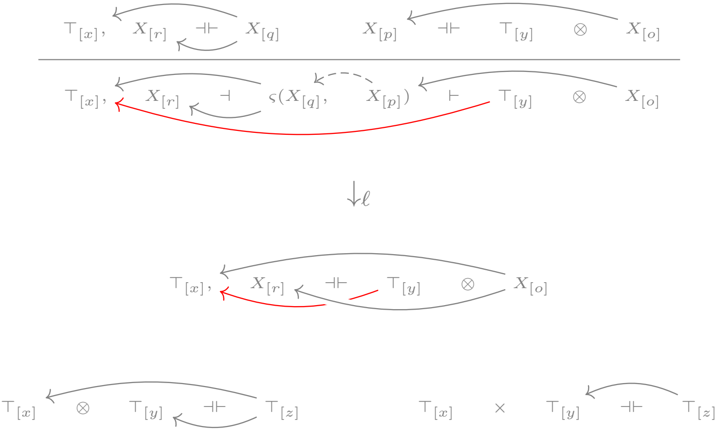

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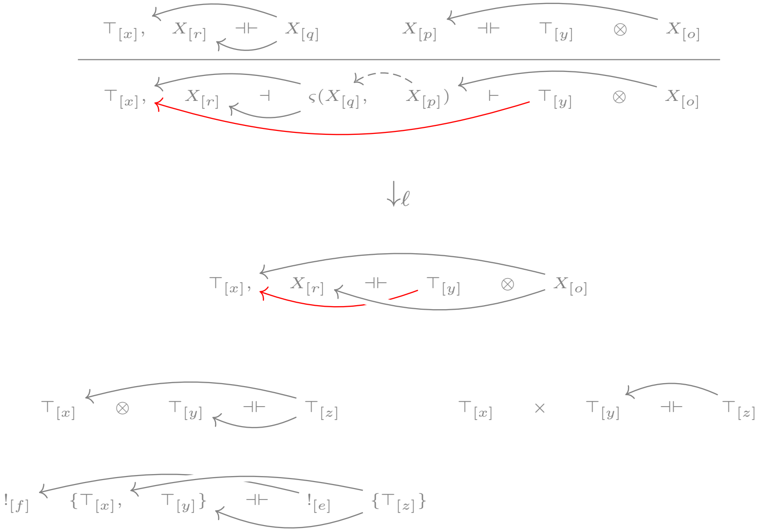

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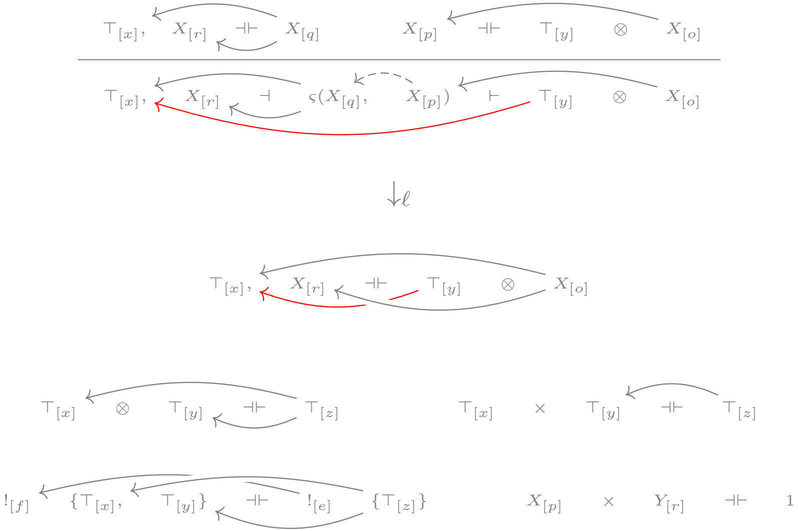
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- **ℓ-types** T – rooted trees – defined by

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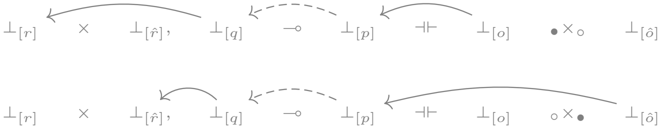

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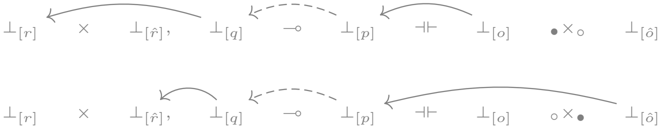

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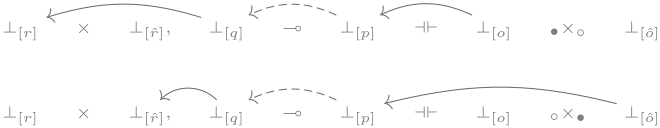

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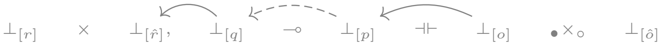

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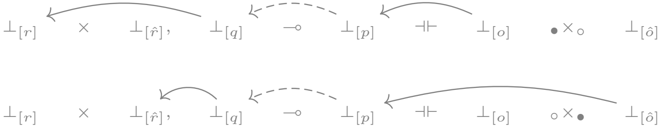

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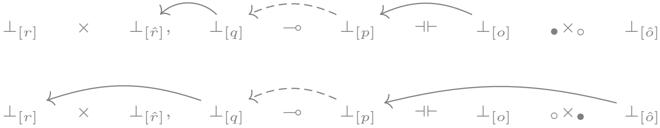

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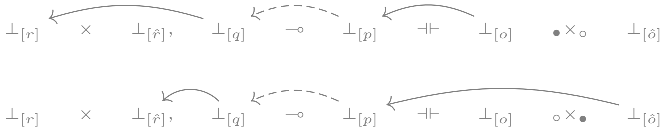

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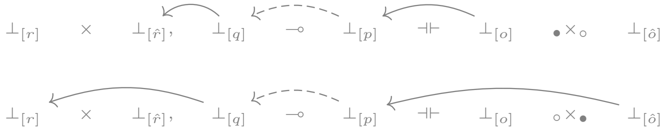
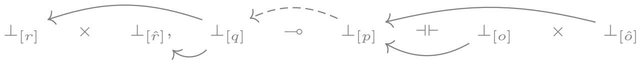


They must be distinguished, but without slicing they would coincide as



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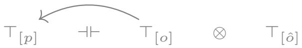

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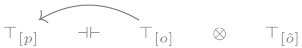

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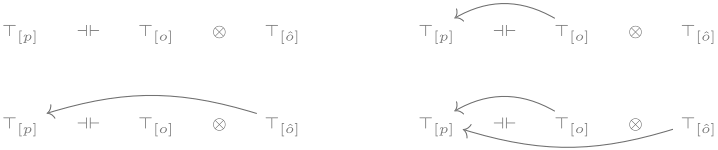

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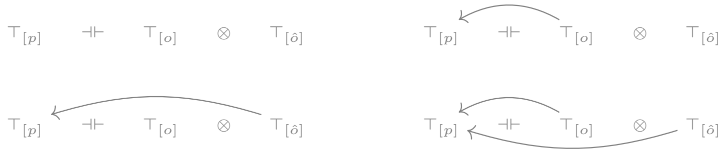

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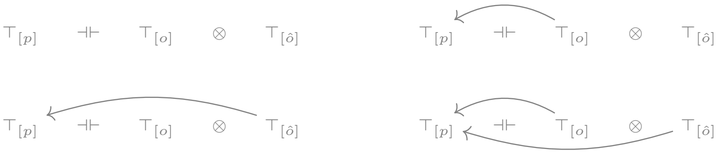

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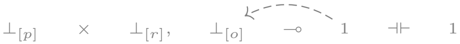

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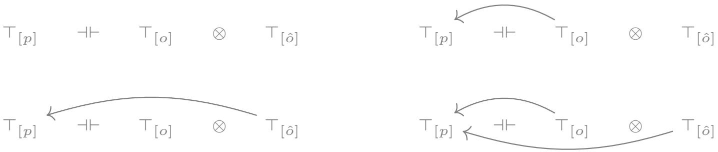

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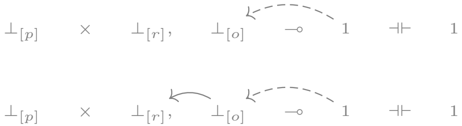

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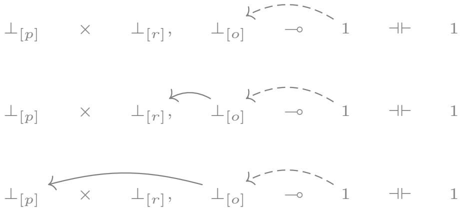

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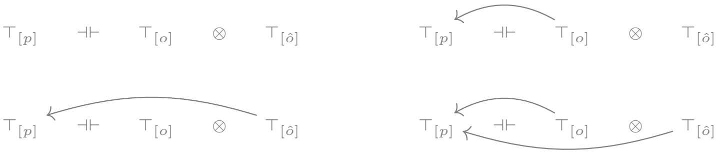

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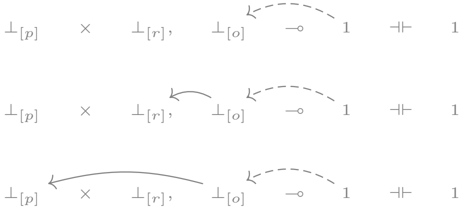

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By this theorem, the **ℓ-equivalence**

$$\tau : F \simeq_\ell \hat{\tau} : \hat{F} :\Leftrightarrow \mathrm{nf}_\ell(\tau : F) = \mathrm{nf}_\ell(\hat{\tau} : \hat{F})$$

between ℓ-graphs is a well-defined equivalence relation.

Plan of the talk

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- The identity $\mathrm{id}_A : A \to A$ links pairs of corresponding leaves.

Plan of the talk

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Corollary (the triple unit problem)

The initial linear category has just one morphism

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and just two morphisms

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Proof.

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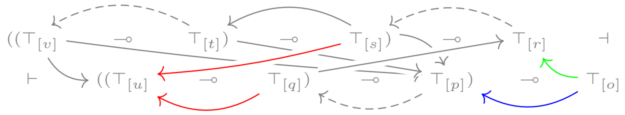

The triple unit problem (2/2)

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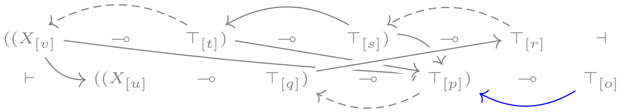
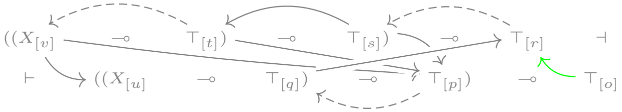
Proof (continued).

For the second part, there are just two ℓ-graphs





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