From Fuzzy Markov Categories Towards Imprecise Probability

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Table of contents

Mathematics of Uncertainty

Deconstructing the Fuzzy Powerset Monad

Deconstructing the Finite Distributions Monad

Summary

Mathematics of Uncertainty

types of uncertainty:

- classical probability distributions over a set X
- quantum probability
- imprecise probability:
  - upper-lower probabilities
  - Dempster-Shafer belief
- fuzzy sets, i.e. functions $X \to [0, 1]$

categorical framework:

- Markov cats [Fri20], probability sheaves [Sim17]
- quantum Markov cats [Par20; FL24]
- models of previsions [Gou24]
- fuzzy powerset monad [Man76]

Imprecise Probability

Joint Work with

- Laura Gonzales Bravo (Madrid)
- Paolo Perrone (Oxford)
- Tomáš Gonda (Innsbruck)

We use Markov cats [Fri20]:

- unifies and generalizes different notions of probability (discrete, continuous, quantum, possibility, . . .)
- abstract, graphical definitions of conditionals, independence, almost sure equality, Bayesian inversion, . . .
- generalizations of theorems (de Finetti, zero-one-laws, strong law of large numbers)

Markov Categories: Overview

symmetric monoidal cats (SMC)

$\cup|$

SMC with projections

$\cup|$

SMC with weak products

$\cup|$

Markov cats

$\cup|$

cartesian monoidal cats

Markov Categories: Example

Kleisli cats!

Example (Finite Distribution Monad)

$$\begin{aligned} \mathsf{D}_{[0,1]} &: \mathsf{Set} \to \mathsf{Set} \\ &X \mapsto \{f : X \to [0,1] \mid \mathrm{supp}(f) < \infty \text{ and } \textstyle\sum_{x \in X} f(x) = 1\} \end{aligned}$$

Fuzzy Powerset Functors

Example (Fuzzy Powerset Monad)

$$\begin{aligned} \mathsf{F}_{[0,1]} : \mathsf{Set} &\to \mathsf{Set} \\ X &\mapsto \{\text{functions } X \to [0,1]\} \end{aligned}$$

Generalizations:

- [Man76] replaces $[0, 1]$ by completely distributive lattices
- we want to replace $[0, 1]$ by quantales

Fuzzy Powerset Functors

Recall:

$$\begin{aligned} \mathsf{F}_{[0,1]} : \mathsf{Set} &\to \mathsf{Set} \\ X &\mapsto \{\text{functions } X \to [0,1]\} \end{aligned}$$

Notation:

$$\begin{aligned} \mathsf{Set}^{\mathsf{op}} \cong \mathsf{CABA} &:= \text{cat. of complete atomic boolean algebras.} \\ \mathsf{SupLat} &:= \text{cat. of suplattices} = \text{cat. of join-complete posets} \end{aligned}$$

Definition

For $L \in \mathsf{SupLat}$

$$\mathsf{F}_L : \begin{array}{ccccc} \mathsf{Set} & \xrightarrow{2^-} & \mathsf{CABA}^{\mathsf{op}} \subseteq \mathsf{SupLat}^{\mathsf{op}} & \xrightarrow{L^-} & \mathsf{Set} \\ X & \longmapsto & 2^X & \longmapsto & \mathsf{SupLat}(2^X, L). \end{array}$$

Towards Fuzzy Powerset Monads

- unit

$$\eta_X : \begin{array}{ccccccc} X & \cong & \mathsf{CABA}(2^X, 2) & \subseteq & \mathsf{SupLat}(2^X, 2) & \xrightarrow{\iota \circ -} & \mathsf{SupLat}(2^X, L) \\ x & \longmapsto & \delta_x & & \longmapsto & & \iota \circ \delta_x \end{array}$$

- multiplication?

Conjecture

For $L \in \mathsf{SupLat}$*:*

$$\{\textit{monad } (F_L, \eta, \mu)\} \overset{\textit{1:1}}{\longleftrightarrow} \{\textit{integral Quantale } (L, \otimes : L \times L \to L)\}\,.$$

Lemma

For $L \in \mathsf{SupLat}$*:*

(F_L, η, μ) *commutative* $\Leftrightarrow \otimes : L \times L \to L$ *commutative*

$\Leftrightarrow$ *Kleisli cat of* F_L *is ~~Markov~~ Copy-Discard cat.*

Intermezzo: Effect Algebras

Definition

An *Effect Algebra* has

- a set E
- constants $0, 1 \in E$
- involution $\neg : E \to E$
- symmetric relation $\mathcal{R} \subseteq E \times E$
- commutative, associative, partial addition $\oplus : \mathcal{R} \to E$

s.th.

$$a\mathcal{R}1 \Leftrightarrow a = 0 \qquad \text{and} \qquad a \oplus b = 1 \Leftrightarrow b = \neg a.$$

Example (Effect Algebras)

- Boolean algebras, e.g. power sets 2^X
- $[0, 1] \subseteq \mathbb{R}$
- $[0, 1] \subseteq$ unital C^*-alg.

Finite Distribution Functor

Goal: monad for $E \in \mathrm{EffAlg}$

$$\mathrm{D}_E : \mathrm{Set} \to \mathrm{Set}$$
$$X \mapsto \{f : X \to E \mid \mathrm{supp}(f) < \infty \text{ and } \textstyle\sum_{x \in X} f(x) = 1\}$$

Notation:

$$\mathrm{D}_E \restriction_{\mathrm{fin}} : \mathrm{FinSet} \to \mathrm{Set} \text{ restriction of } \mathrm{D}_E$$
$$\mathrm{FinSet}^{\mathrm{op}} \cong \mathrm{FinCABA} := \text{cat. of finite CABAs}$$
$$\mathrm{EffAlg} := \text{cat. of effect algebras}$$

Definition

For $E \in \mathrm{EffAlg}$

$$\mathrm{D}_E \restriction_{\mathrm{fin}} : \begin{array}{ccccc} \mathrm{FinSet} & \xrightarrow{2^-} & \mathrm{FinCABA}^{\mathrm{op}} \subseteq \mathrm{EffAlg}^{\mathrm{op}} & \xrightarrow{E^-} & \mathrm{Set} \\ X & \longmapsto & 2^X & \longmapsto & \mathrm{EffAlg}(2^X, E). \end{array}$$

Relative Monads

...consist of

- functors

$$\begin{array}{ccc} \mathsf{FinSet} & \xrightarrow{T} & \mathsf{Set} \\ {\scriptstyle J}\big\downarrow & & \\ \mathsf{Set} & & \end{array}$$

- unit (for $X \in \mathsf{FinSet}$):

$$\eta_X : JX \to TX$$

- 'Kleisli' extension of $f : JX \to TY$ (for $X, Y \in \mathsf{FinSet}$):

$$f^\sharp : TX \to TY$$

similar to Kleisli extension [ACU15].
[ACU15] constructs monad

$$\begin{array}{ccc} \mathsf{FinSet} & \xrightarrow{T} & \mathsf{Set} \\ {\scriptstyle J}\big\downarrow & \overset{\mathsf{Lan}}{\dashrightarrow} & \\ \mathsf{Set} & & \end{array}$$

Towards Finite Distribution Monads

Goal: relative monad on

$$T = \mathrm{D}_E \restriction_{\mathrm{fin}} : \begin{array}{ccccc} \mathrm{FinSet} & \xrightarrow{2^-} & \mathrm{FinCABA}^{\mathrm{op}} \subseteq \mathrm{EffAlg}^{\mathrm{op}} & \xrightarrow{E^-} & \mathrm{Set} \\ X & \longmapsto & 2^X & \longmapsto & \mathrm{EffAlg}(2^X, E). \end{array}$$

- unit

$$\eta_X : \begin{array}{ccccccc} X & \cong & \mathrm{CABA}(2^X, 2) & \subseteq & \mathrm{EffAlg}(2^X, 2) & \xrightarrow{\iota \circ -} & \mathrm{EffAlg}(2^X, L) \\ x & \longmapsto & \delta_x & & \longmapsto & & \iota \circ \delta_x \end{array}$$

- 'Kleisli' extension?

Conjecture

For $E \in \mathrm{EffAlg}$:

$$\{\textit{relative monad } (D_E \restriction_{fin}, \eta, \sharp)\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} m : E \times E \to E \\ \textit{associat., distributive, unital} \end{array} \right\}.$$

Towards Finite Distribution Monads II

Assume for $E \in \mathsf{EffAlg}$:

$$\{\text{relative monad } (\mathsf{D}_E \restriction_{\mathsf{fin}}, \eta, \sharp)\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} m : E \times E \to E \\ \text{associat., distributive, unital} \end{matrix} \right\}.$$

Lemma

For $E \in \mathsf{EffAlg}$:

$$\begin{aligned} \textit{Lan commutative} &\Leftrightarrow m : E \times E \to E \textit{ commutative} \\ &\Leftrightarrow \textit{Kleisli cat of Lan is Markov.} \end{aligned}$$

Example

- $E = [0,1] \subseteq \mathbb{R}$: model classical finite probability
- $E = \{0,1\}$: possibility (non-empty finite powerset monad)
- $E = [0,1] \subseteq$ unital C^*-alg: quantum probabilistic processes

Summary

- fuzzy powerset and finite distributions have similar shape

$$\mathsf{F}_{[0,1]} : \mathsf{Set} \xrightarrow{2^{-}} \mathsf{SupLat}^{\mathsf{op}} \xrightarrow{[0,1]^{-}} \mathsf{Set}$$

$$\mathsf{D}_{[0,1]} \restriction_{\mathsf{fin}} : \mathsf{FinSet} \xrightarrow{2^{-}} \mathsf{EffAlg}^{\mathsf{op}} \xrightarrow{[0,1]^{-}} \mathsf{Set}.$$

- generalizations to other truth values than $[0,1]$:
  - join-complete posets L for fuzzy set monad F_L
  - effect algebras E for relative distribution monad D_E

Future work:

- new notion of quantum Markov cats.
- continuous case (Giry monad)
- useful for imprecise probability?

Obrigado.

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