The Lawvere Condition and a unification of Malt'sev-like categories

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Ah! ser de Aveiro, como este sal!

1 - Multiplicative Graphs and Preorders

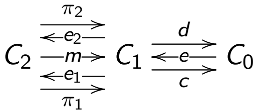
A **multiplicative graph** generalizes the concept of a reflexive relation, or more precisely, a preorder.

If the pair $(d, c) : C_1 \to C_0 \times C_0$ is *jointly monic*, the multiplication m becomes **unique**, and the structure corresponds to an **internal preorder**.

In general, however, the multiplication need not be unique. We then consider **unital multiplicative graphs**, where m satisfies the two extra conditions:

$$m \circ e_1 = 1_{C_1}, \quad m \circ e_2 = 1_{C_1}$$

Multiplicative graph



(d, e, c) reflexive graph

$dm = d\pi_2, \; cm = c\pi_1$

(π_1, e_1, π_2, e_2) local product

Structure morphisms and conditions for a multiplicative graph internal to an arbitrary category $\mathcal{C}$.

2 - The Lawvere Condition

The original **Lawvere condition** states that the forgetful functor from **internal groupoids** to **reflexive graphs** is an isomorphism.

However, this is quite a strong requirement. An equivalent but *much weaker* formulation says:

> *The forgetful functor from* **unital multiplicative graphs** *to* **reflexive graphs** *admits a section.*

This expresses a similar principle as in the case of preorders: the multiplication is uniquely determined by the underlying reflexive structure.

Grpd $\to$ **RG** is an iso
Cat $\to$ **RG** is an iso
UMG $\to$ **RG** is an iso
Grpd $\to$ **RG** has a section
Cat $\to$ **RG** has a section
UMG $\to$ **RG** has a section

The above condtions are equivalent in any category with local products:

$$A \underset{\langle 1_A, sf\rangle}{\overset{\pi_1}{\leftrightarrows}} A \times_B C \underset{\langle rg, 1_C\rangle}{\overset{\pi_2}{\rightleftarrows}} C$$

$$fr = 1_B = gs$$

$$A \underset{r}{\overset{f}{\rightleftarrows}} B \underset{s}{\overset{g}{\leftrightarrows}} C$$

3 - The Goal: A Unified Framework

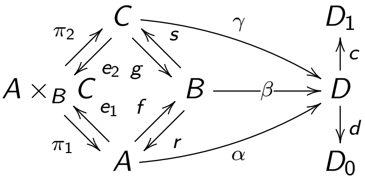
The goal of this presentation is to provide a **unifying perspective** on three important classes of categories:

- **Naturally Mal'tsev** categories
- **Mal'tsev** categories
- **Weakly Mal'tsev** categories

Which are defined, respectively, as:

- The Lawvere Condition holds true
- Every reflexive relation is a tolerance relation (reflexive and symmetric).
- every local product injection cospan is jointly epimorphic.

The $\mathcal{M}$-Kite Condition provides a unified structural framework generalizing internal categories and pregroupoids.

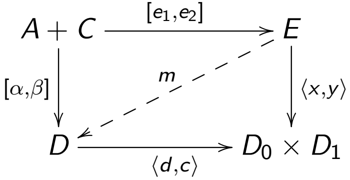


If the span (d, c) lies in a class $\mathcal{M}$ and $fr = 1_B = gs$, $\alpha r = \beta = \gamma s$, $d\alpha = d\beta f$, $c\gamma = c\beta g$, then there exists a unique $m\colon A \times_B C \to D$ such that $me_1 = \alpha$, $me_2 = \gamma$, $dm = d\gamma\pi_2$, $cm = c\alpha\pi_1$.

4 - The Three Classes via Spans

Each class of categories corresponds to a different class of spans $\mathcal{M}$, which must consist of spans whose legs have kernel pairs, are closed under the kernel pair construction, and include all local products.

- **Naturally Mal'tsev:** all spans (whose legs have kernel pairs)
- **Mal'tsev:** all jointly monomorphic spans (relations) whose legs have kernel pairs
- **Weakly Mal'tsev:** all jointly strongly monomorphic spans (strong relations) whose legs have kernel pairs.

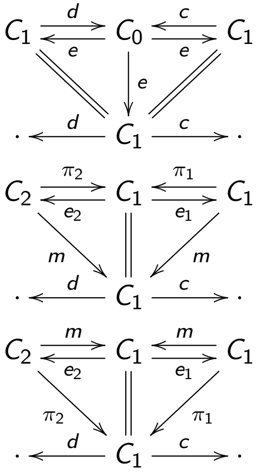


A strong relation is a span (d, c) orthogonal to every jointly epic cospan (e_1, e_2). The relevant notion here is a special case depending on commuting split spans (p_1, e_1, p_2, e_2), not just on cospans, where $x = d\gamma p_2$ and $y = c\alpha p_1$.

5 - The Unifying Theorem

TFAE on any category with local products, with $\mathcal{M}$ as before:

Grpd$(\mathcal{M}) \to$ **RG**$(\mathcal{M})$ is an iso
Cat$(\mathcal{M}) \to$ **RG**$(\mathcal{M})$ is an iso
UMG$(\mathcal{M}) \to$ **RG**$(\mathcal{M})$ is an iso
AssPreGrpd$(\mathcal{M}) \to$ **Span**$(\mathcal{M})$ is an iso
PreGrpd$(\mathcal{M}) \to$ **Span**$(\mathcal{M})$ is an iso
Grpd$(\mathcal{M}) \to$ **RG**$(\mathcal{M})$ has a section
Cat$(\mathcal{M}) \to$ **RG**$(\mathcal{M})$ has a section
UMG$(\mathcal{M}) \to$ **RG**$(\mathcal{M})$ has a section
AssPreGrpd$(\mathcal{M}) \to$ **Span**$(\mathcal{M})$ has section
PreGrpd$(\mathcal{M}) \to$ **Span**$(\mathcal{M})$ has a section
Every local product is $\mathcal{M}$-compatible
The $\mathcal{M}$-Kite Condition holds
Every dikite with direction in $\mathcal{M}$ is multiplicative (in a unique way).



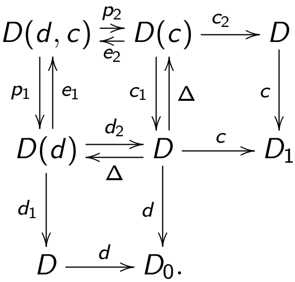
A **Kock pregroupoid** is a span equipped with a partial composition $p\colon D(d,c) \to D$ satisfying coherence axioms that generalize difunctional relations.
For all $x, y, z\colon Z \to D$ with $dx = dy$ and $cy = cz$, the following hold:

$$dp\langle x,y,z\rangle = dz, \quad cp\langle x,y,z\rangle = cx,$$
$$p\langle x,y,y\rangle = x, \quad p\langle x,x,y\rangle = y.$$

The associativity-like axiom also holds:

$$p\langle p\langle u,v,x\rangle, y, z\rangle = p\langle u, v, p\langle x,y,z\rangle\rangle.$$

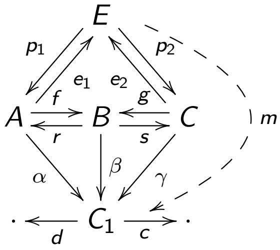
As in the Lawvere condition, the forgetful functor from Kock pregroupoids to spans on a given class $\mathcal{M}$ admits a *section*.



When (d,c) is a relation, this says that every such relation is difunctional.

The $\mathcal{M}$-**Kite condition** is a structural lifting property involving a span $(d, c) \in \mathcal{M}$ and a *commuting split span*, defined as a quadruple (p_1, e_1, p_2, e_2) satisfying $p_1 e_1 = 1_A$, $p_2 e_2 = 1_C$ and $e_1 p_1 e_2 p_2 = e_2 p_2 e_1 p_1$.

For every diagram as shown, with (p_1, e_1, p_2, e_2) a commuting split span, (p_1, p_2) jointly monic, $(p_1, p_2), (d, c) \in \mathcal{M}$, $\alpha(p_1 e_2 p_2) = \gamma(p_2 e_1 p_1)$, $d\alpha p_1 = d\alpha p_1 e_2 p_2$, $c\gamma p_2 = c\gamma p_2 e_1 p_1$
the $\mathcal{M}$-Kite condition asserts that there exists a *unique* morphism $m \colon E \to D$ such that: $me_1 = \alpha$, $me_2 = \gamma$, $dm = d\gamma p_2$, $cm = c\alpha p_1$.



The $\mathcal{M}$-Kite condition expresses orthogonality between a span in $\mathcal{M}$ and a commuting split span, like a local product, without requiring the splittings (f, r) and (g, s).

8 – Why this level of generality?

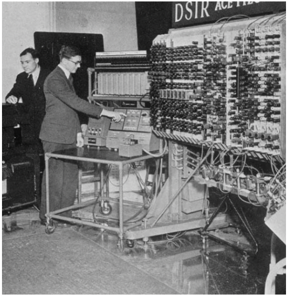
Categories with limited pullbacks—such as those admitting only pullbacks along monomorphisms—arise naturally in both mathematics and applications.

A classical example is the category of differentiable manifolds, which admits pullbacks along smooth embeddings, but not all pullbacks.

Another example is the category where:

- Objects: $0, 1, \ldots, 2^{64}$.
- Morphisms $n \to m$ are pairs (m, u), $u = (u_1, \ldots, u_n)$, $u_i \leq m$.
- Composition: $(m, u) \circ (\text{numel(u)}, v)$ is $(m, u(v))$, in Matlab notation.

The Cosmic Paradox: Is the Universe Finite or Infinite?



Models array indexing in `Matlab` and `Octave`. Pullbacks along monos exist, but general pullbacks do not.

9 – Pullbacks and index operations in programming

This category arises naturally in the semantics of array-based programming languages like `Matlab` and `Octave`.

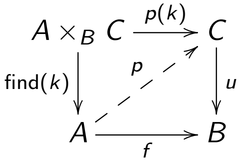
For example, consider the function:

```
[k, p] = ismember(f, u)
```

where:

- u is a vector with unique entries,
- f is a general index vector.

The output pair $(\texttt{find}(k), p(k))$ corresponds to projections of the pullback of f along u in the above category.



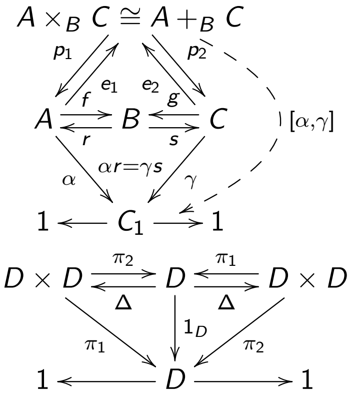
This justifies studying category-theoretic conditions like the $\mathcal{M}$-Kite condition without assuming full limits, focusing instead on local products arising from split epis over the same base object.

Let $\mathbb{C}$ be a category with local products.

Main result: $\mathbb{C}$ is of *naturally Mal'tsev type* if and only if the equivalent conditions of The Unifying Theorem hold for the class $\mathcal{M}_0$ of all spans whose legs have kernel pairs.

If assuming binary products and a terminal object then:

- every object in $\mathbb{C}$ admits a *canonical Mal'tsev operation*.
- every local product is also a local coproduct.



The Kite condition gives the naturally Malt'sev operation $p \colon D \times D \times D \to D$.

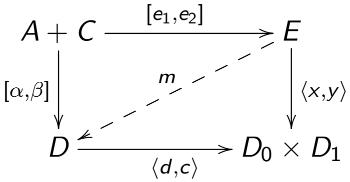
11 – The Mal'tsev Case

Let $\mathbb{C}$ be a category with local products.

Main result: $\mathbb{C}$ is of *Mal'tsev type* if and only if the equivalent conditions of the Unifying Theorem hold for the class $\mathcal{M}_1$, consisting of jointly monomorphic spans whose legs admit kernel pairs.

If $\mathbb{C}$ has binary products and pullbacks along monomorphisms, then:

- every local product injection cospan (e_1, e_2) is jointly extremally epimorphic.



In general, we can only say that (e_1, e_2), as part of a local product, is compatible with every jointly monic span (d, c). This defines an intermediate notion between extremal and strong epimorphism, where we require $x = d\gamma p_2$ and $y = c\alpha p_1$.

12 – The Weakly Mal'tsev Case

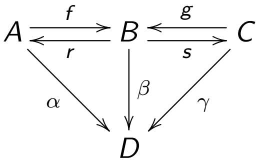
Let $\mathbb{C}$ be a category with local products.

Main result: $\mathbb{C}$ is of *weakly Mal'tsev type* if and only if the equivalent conditions of the Unifying Theorem hold for the class $\mathcal{M}_2$: spans whose legs admit kernel pairs and are compatible with jointly epimorphic commuting split spans.

If $\mathbb{C}$ has finite limits, then $\mathcal{M}_2$ coincides with the class of *strong relations*.

As before, a span in $\mathcal{M}_2$ is generally not orthogonal to the class of all epimorphisms, since we must restrict to $x = d\gamma p_2$ and $y = c\alpha p_1$ in the previous orthogonality square.

The original kite diagram



$fr = 1_B = gs$, $\alpha r = \beta = \gamma s$.
There exists at most one $m\colon A \times_B C \to D$ with $me_1 = \alpha$ and $me_2 = \gamma$.
This can be used to define weakly Mal'tsev objects in arbitrary categories.

13 – References

1 D. Bourn, M. Gran and P.-A. Jacqmin *On the naturalness of Mal'tsev categories*, 2021.

2 A. Carboni, G. M. Kelly, and R. M. Pedicchio. *Some remarks on Malt'sev and Goursat categories*, 1993.

3 Z. Janelidze and NMF. *Weakly Mal'tsev Categories and Strong Relations*, 2012.

4 NMF. *On Naturally and Weakly Mal'tsev Categories.* arXiv:2508.13315, 2025.

5 NMF and T. Van der Linden, *Categories vs. Groupoids via Generalised Mal'tsev Properties*, 2014.

All references are available online.

Thank you for your attention.