Induction for extended affine type A Soergel bimodules from a maximal parabolic

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Outline

- Brief history
- Categorical induction: the general idea
- Extended affine type A Soergel bimodules
- Induction from a maximal parabolic

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- **Wide finitary generalization**: Macpherson, 2022.
- **Triangulated birepresentations of extended affine type A Soergel bimodules**:
  - Evaluation birepresentations (M.-Miemietz-Vaz, 2024).
  - Induction from maximal parabolics (M.-Miemietz-Vaz, arXiv:2507.02347).

Categorical induction: the general idea

- Representation: A - f.d. $\Bbbk$-algebra, M - f.d. $\Bbbk$-vector space and a homomorphism of $\Bbbk$-algebras

$$\mathbf{M}\colon A \to \mathrm{End}_{\Bbbk}(M)$$

Induction and restriction

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- Induction: $\mathrm{Ind}_B^A(M) := A \otimes_B M$

Example: parabolic induction

- Maximal parabolic: $n = k + m$,

$$\begin{array}{rcl} \mathbb{C}[S_k] \otimes_{\mathbb{C}} \mathbb{C}[S_m] & \hookrightarrow & \mathbb{C}[S_n] \\ s_i \otimes 1 & \mapsto & s_i \quad (i = 1, \ldots, k-1) \\ 1 \otimes s_j & \mapsto & s_{k+j} \quad (j = 1, \ldots, m-1) \end{array}$$

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- General case: $n = k_1 + \cdots + k_r$,

$$\mathbb{C}[S_{k_1}] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[S_{k_r}] \hookrightarrow \mathbb{C}[S_n]$$

Categorical induction and restriction

- Birepresentation: $\mathcal{A}$ - finitary monoidal category, $\mathcal{M}$ - finitary category and a linear monoidal functor

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- Right X-modules form a left $\mathcal{C}$-birepresentation:

$$\begin{aligned} \mathcal{C} \boxtimes \mathrm{mod}_{\mathcal{C}}(\mathrm{X}) &\to \mathrm{mod}_{\mathcal{C}}(\mathrm{X}) \\ \mathrm{F} \boxtimes \mathrm{M} &\mapsto \mathrm{F} \circ \mathrm{M} \end{aligned}$$

Example

▶Dual numbers: Let $D := \mathbb{C}[x]/(x^2)$. Note that D is a Frobenius algebra with counit (trace) and comultiplication

$$\epsilon_D \colon D \to \mathbb{C}, \qquad \delta_D \colon D \to D \otimes_{\mathbb{C}} D$$
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▶Monoidal structure:

$$\begin{aligned}(D \otimes_{\mathbb{C}} D) \otimes_D (D \otimes_{\mathbb{C}} D) &\cong D \otimes_{\mathbb{C}} (D \otimes_D D) \otimes_{\mathbb{C}} D\\ &\cong D \otimes_{\mathbb{C}} D \otimes_{\mathbb{C}} D\\ &\cong (D \otimes_{\mathbb{C}} D) \oplus (D \otimes_{\mathbb{C}} D) \in \mathcal{S}.\end{aligned}$$

Thus $\mathcal{S} = (\mathcal{S}, \otimes_D, D)$ is a finitary monoidal category.

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▶Algebra object 1: D is an algebra object in $\mathcal{S}$, with

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$$\begin{aligned} \mu_{\mathrm{B}}\colon (D \otimes_{\mathbb{C}} D) \otimes_D (D \otimes_{\mathbb{C}} D) &\to D \otimes_{\mathbb{C}} D \\ (f \otimes g) \otimes (h \otimes j) &\mapsto \epsilon_D(gh) f \otimes j \end{aligned}$$

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►$\mathcal{S}$-Birepresentations: We have

$$\mathrm{mod}_{\mathcal{S}}(D) \simeq \mathcal{S} \quad \text{and} \quad \mathrm{mod}_{\mathcal{S}}(\mathrm{B}) \simeq D\text{-pmod}.$$

Categorical induction using algebra objects

▶ Internal hom-construction:

$$\begin{array}{ccc} \{\text{Left finitary } \mathcal{C}\text{-bireps}\} & \text{"} \longrightarrow \text{"} & \{\text{Algebra objects in } \mathcal{C}\} \\ \mathbf{M} & \mapsto & \mathrm{X}_{\mathbf{M}} \end{array}$$

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▶ Categorical induction: Let $\Psi\colon \mathcal{B} \to \mathcal{A}$ be a $\Bbbk$-linear monoidal functor/embedding. Define

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- The internal hom-construction is not well understood yet.
- The algebra objects are infinite countable coproducts of indecomposable objects.
- There is no general theorem saying that $\mathrm{mod}_{K^b(\mathcal{A})}(\Psi(\mathrm{B}))$ is triangulated.

Extended affine type A Soergel bimodules

Diagrammatic calculus for $\mathcal{S}$

Recall $D = \mathbb{C}[x]/(x^2)$, $\mathcal{S} = \mathrm{add}(D \oplus \mathrm{B})$, with $\mathrm{B} := D \otimes_{\mathbb{C}} D$.

- **Generators**:

$\mathrm{id}_{\mathrm{B}} :=$ 

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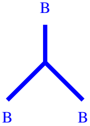
$\mu_{\mathrm{B}} :=$ 

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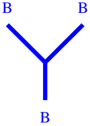
$\delta_{\mathrm{B}} :=$ 

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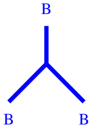
$\mu_{\mathrm{B}} :=$ 

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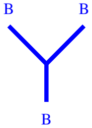
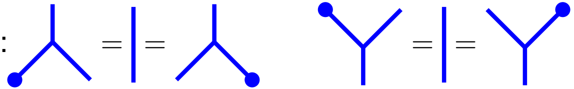
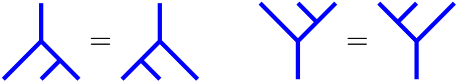
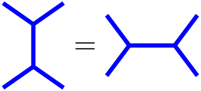
$\delta_{\mathrm{B}} :=$ 

- **Relations**:

(Co)unitality: 

(Co)associativity: 

Frobenius rels: e.g. 

- **Composition**: stacking (vertical).
- **Monoidal product**: juxtaposition (horizontal).

Diagrammatic calculus for $\mathcal{S}$

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- **Monoidal product**: juxtaposition (horizontal).
- **Linearity**: The diagrammatic category $\mathcal{BS}$ is a $\mathbb{C}$-linear and monoidal.
- **Envelopes**: $\mathcal{S} \cong \mathrm{Kar}(\mathrm{Mat}(\mathcal{BS}))$.

Soergel calculus for extended affine type A_{n-1}

Elias-Khovanov, Elias, M.-Thiel: The $\mathbb{R}$-linear monoidal category of $\widehat{\mathcal{BS}}_n^{\mathrm{ext}}$ of Bott-Samelson bimodules:

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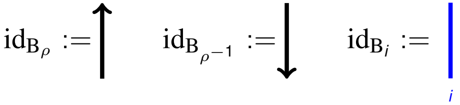
- Morphisms: Identity morphisms



Soergel calculus for extended affine type A_{n-1}

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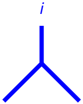
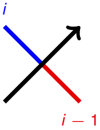
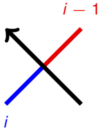
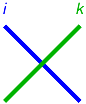
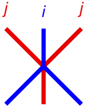






$n \geq 3$: 

$|i-k|>1$ 

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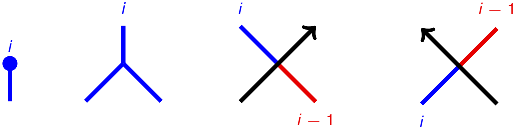

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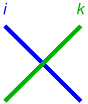

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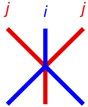

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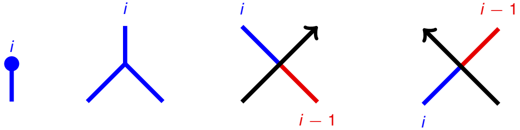

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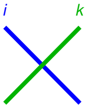

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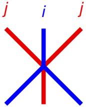

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- Soergel bimodules: $\widehat{\mathcal{S}}_n^{\mathrm{ext}} := \mathrm{Kar}(\mathrm{Mat}(\widehat{\mathcal{BS}}_n^{\mathrm{ext}}))$.
- Categorification theorem: $[\widehat{\mathcal{S}}_n^{\mathrm{ext}}]_{\oplus} \cong \widehat{H}_n^{ext}$.

- Rouquier complexes: Define $\mathrm{T}_i^{\pm 1} \in K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$:

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for $i = 0, \ldots, n-1$.

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- **Good news**: Partial calculus suffices. DIY!

Induction from a maximal parabolic

A symmetric pair of monoidal functors

- Embeddings:

$$\Psi_L \colon \widehat{\mathcal{BS}}_k^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}}) \quad \text{and} \quad \Psi_R \colon \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$$

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- Symmetric braiding: $\Psi_L \Psi_R \cong \Psi_R \Psi_L$.

A symmetric pair of monoidal functors

- Embeddings:

$$\Psi_L \colon \widehat{\mathcal{BS}}_k^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}}) \quad \text{and} \quad \Psi_R \colon \widehat{\mathcal{BS}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$$

- Symmetric braiding: $\Psi_L \Psi_R \cong \Psi_R \Psi_L$.

Theorem (M.-Miemietz-Vaz)

The symmetric pair Ψ_L, Ψ_R induces a linear monoidal functor

$$\Psi_{k,n-k} \colon \widehat{\mathcal{S}}_k^{\mathrm{ext}} \boxtimes \widehat{\mathcal{S}}_{n-k}^{\mathrm{ext}} \to K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}})$$

such that

$$\Psi_{k,n-k}(X \boxtimes Y) := \Psi_L(X)\Psi_R(Y)$$
$$\Psi_{k,n-k}(f \boxtimes g) := \Psi_L(f)\Psi_R(g),$$

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- Induction: $\mathrm{Y} := \Psi_{1,1}(\mathrm{X} \boxtimes \mathrm{X}) \in K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext},\diamond})$, where

$$\mathrm{Y} \cong \coprod_{r \in \mathbb{Z}} \coprod_{s \in \mathbb{Z}} (\mathrm{B}_\rho \mathrm{T}_1)^r (\mathrm{T}_1^{-1} \mathrm{B}_\rho)^s.$$

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- Induced triangulated birepresentation: By a general construction due to Fan–Keller–Qiu, there is a triangulated closure of

$$\mathrm{add}\left\{ \mathrm{FY} \mid \mathrm{F} \in K^b(\widehat{\mathcal{S}}_n^{\mathrm{ext}}) \right\} \subset K^b(\widehat{\mathcal{S}}_2^{\mathrm{ext},\diamond}).$$

The end

Thanks!