XV Portuguese Category Seminar
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Indexed monoidal structures & regular double hyperdoctrines

(José Siqueira, Topos Institute, Oxford, UK)

based on 'Double-functorial representation of indexed monoidal structures'

arXiv: 2508.06637

see Dawson, Paré, Pronk (2010)

Idea: $\mathrm{Span}(\mathcal{C})^{op}$ represents Beck-Chevalley fibrations $\mathcal{C}^{op} \xrightarrow{P} K$. What about those also satisfying Frobenius reciprocity?

cartesian, but may not have B-C property for all pullbacks

cartesian 2-cat.

Thm: A (generalised) regular hyperdoctrine $P: \mathcal{C}^{op} \longrightarrow K$

pseudofunctorial

is the same as a lax symmetric monoidal double pseudofunctor

$$\mathrm{Span}(\mathcal{C})^{op} \xrightarrow{P^{\bullet}} \mathbb{Q}\mathrm{t}(K)$$

with companion commuter laxators.

↪ adapted from Lawvere 1969

Def: Let $\mathcal{C}$ be a cartesian category. A regular hyperdoctrine over $\mathcal{C}$ is a functor $\mathcal{C}^{op} \xrightarrow{P} \mathrm{Pos}$ such that:

i.e., a cartesian monoidal poset

- Each poset PX is a $\wedge$-semilattice;
- Each $PY \xrightarrow{Pg} PX$ (for $X \xrightarrow{g} Y$ in $\mathcal{C}$) has a left adjoint $PX \xrightarrow{\exists g} PY$;
- For any pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "k"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

in $\mathcal{C}$, we have

$$\exists f \circ Ph = Pk \circ \exists g$$

(Beck-Chevalley)

- For any $X \xrightarrow{g} Y$ in $\mathcal{C}$, $\varphi \in PX$, and $\psi \in PY$ we have $\exists g(Pg(\psi) \wedge \varphi) = \psi \wedge \exists g(\varphi)$

(Frobenius reciprocity)

⇝ adapted from Barwick 2017, Haugseng et al 2020

Def: A cartesian adequate triple $(\mathcal{C}, L, R)$ consists of a cartesian category $\mathcal{C}$ and classes $L, R \subseteq \mathrm{mor}(\mathcal{C})$ such that:

- L and R are closed under composition, finite products, projections, and identities;
- Pullbacks

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "k"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

$k \rightsquigarrow k \in R$

$g \rightsquigarrow \in L$

exist in $\mathcal{C}$, and we have $h \in R$, $f \in L$.

This is what is needed to compose spans

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[dr, "x_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

$\in L \rightsquigarrow x_1$

$x_2 \rightsquigarrow \in R$

Def: Let $(\mathcal{C}, \mathcal{L}, \mathcal{R})$ be a cart. adeq. triple and $\mathcal{K}$ be a cartesian 2-category. A $(\mathcal{C}, \mathcal{L}, \mathcal{R})$-regular hyperdoctrine with semantics in $\mathcal{K}$ is a pseudo functor $P : \mathcal{C}^{op} \longrightarrow \mathcal{K}$ such that:

- Each poset PX is a pseudomonoid in $\mathcal{K}$;
- Each $PY \xrightarrow{Pg} PX$ (for $X \xrightarrow{g} Y$ in $\mathcal{R}$) has a left adjoint $PX \xrightarrow{\exists g} PY$;
- For any pullback

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "k \in \mathcal{R}"] \\ X \arrow[r, "g \in \mathcal{L}"'] & Y \end{tikzcd}$$

in $\mathcal{C}$, the canonical map $\exists f \, Ph \Rightarrow Pk \, \exists g$ is invertible (B.C);

- For any $X \xrightarrow{g} Y$ in $\mathcal{R}$, the canonical map

$$\exists g \,(Pg \otimes_{PX} id_{PX}) \longrightarrow \exists g \, Pg \otimes_{PY} \exists g \longrightarrow id_{PY} \otimes_{PY} \exists g$$

is invertible (Frobenius reciprocity).

Examples:

- $(\mathcal{C}, \text{all}, \text{all}) \rightsquigarrow$ traditional regular hyperdoctrines

  $\rightsquigarrow \mathcal{C}^{op} \xrightarrow{P} \text{Cat}$ $\rightsquigarrow \exists_f$ = dependent sum

  $X \mapsto \mathcal{C}/X$

  $\rightsquigarrow \mathcal{C} = \text{Set}$, $\text{Set}^{op} \xrightarrow{P} \text{Pos}$

  $X \mapsto [0,\infty]^X$

  ↳ Lawvere quantale

  i.e., $\mathcal{C}$ has finite limits

  generates lenses by taking Span

- L = vertical maps, R = cartesian maps for cart. fibration $\pi : \mathcal{C} \to \mathcal{B}$;

- $\mathcal{C}$ = Top := compact. gen. spaces, L = Serre fibrations, R = all

$P : X \longmapsto \text{Ho}(\text{Top} \downarrow X)$ is $(\mathcal{C}, L, R)$-regular with semantics in Cat

c.f. Shulman 2011

- $\mathcal{C}$ = LKHaus := loc. compact Hausdorff spaces, L = all, R = proper maps

$P : X \mapsto \text{Ho}\left(\begin{array}{l}\text{unbounded chain} \\ \text{complexes of sheaves} \\ \text{of abelian groups over} \\ X\end{array}\right)$ is $(\mathcal{C}, L, R)$ (co)regular.

We work with (weak/pseudo) double categories

$$\begin{tikzcd}
A_1 \arrow[r, "A", "|" marking] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & A_2 \arrow[r, "A'", "|" marking] \arrow[d, "f_2"] \arrow[dr, phantom, "\beta"] & A_3 \arrow[d, "f_3"] \\
B_1 \arrow[r, "B"', "|" marking] \arrow[d, "g_1"'] \arrow[dr, phantom, "\gamma"] & B_2 \arrow[r, "B'"', "|" marking] \arrow[d, "g_2"] \arrow[dr, phantom, "\delta"] & B_3 \arrow[d, "g_3"] \\
C_1 \arrow[r, "C"', "|" marking] & C_2 \arrow[r, "C'"', "|" marking] & C_3
\end{tikzcd}$$

$$\left(\frac{\alpha}{\beta}\right) \Bigg| \left(\frac{\beta}{\delta}\right) = \frac{(\alpha|\beta)}{(\gamma|\delta)}$$

Recap: $A \overset{f^*}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B$ is a companion of

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$$

if $\exists$
$$\begin{tikzcd} A \arrow[r, equal, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner_f"] & A \arrow[d, "f"] \\ A \arrow[r, "f^*"', "|" marking] & B \end{tikzcd}$$
and
$$\begin{tikzcd} A \arrow[r, "f^*", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "f_\lrcorner"] & B \arrow[d, equal] \\ B \arrow[r, equal, "|" marking] & B \end{tikzcd}$$
such that

$$\begin{tikzcd} A \arrow[r, "f^*", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_{f'}"] & B \arrow[d, equal] \\ A \arrow[r, "f^*"', "|" marking] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, equal, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner_f"] & A \arrow[r, "f^*", "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "f_\lrcorner"] & B \arrow[d, equal] \\ A \arrow[r, "f^*"', "|" marking] & B \arrow[r, equal, "|" marking] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, equal, "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{id}"] & A \arrow[d, "f"] \\ B \arrow[r, equal, "|" marking] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, equal, "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner_f"] & A \arrow[d, "f"] \\ A \arrow[r, "f^*"', "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "f_\lrcorner"] & B \arrow[d, equal] \\ B \arrow[r, equal, "|" marking] & B \end{tikzcd}$$

Dually, we speak of a conjoint $B \xrightarrow{f!} A$.

Def: A square

$$\begin{tikzcd}
A_1 \arrow[r, "A", "\shortmid" marking] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & A_2 \arrow[d, "f_2"] \\
B_1 \arrow[r, "B"', "\shortmid" marking] & B_2
\end{tikzcd}$$

is a

$\rightsquigarrow$ Paré 2024

companion commuter if f_1 and f_2 have companions and

$$\begin{tikzcd}
A_1 \arrow[r, equal, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner f"] & A_1 \arrow[r, "A", "\shortmid" marking] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha"] & A_2 \arrow[r, "f_2^*", "\shortmid" marking] \arrow[d, "f_2"] \arrow[dr, phantom, "f_2 \lrcorner"] & B_2 \arrow[d, equal] \\
A_1 \arrow[r, "f_1^*"'] & B_1 \arrow[r, "B"', "\shortmid" marking] & B_2 \arrow[r, equal, "\shortmid" marking] & B_2
\end{tikzcd}$$

is invertible.

A tight transformation $\lambda : F \to G$ is a **companion commuter transformation** if its components

$$\begin{tikzcd}
FX_1 \arrow[r, "FX", "|" marking] \arrow[d, "\lambda_{X_1}"'] \arrow[dr, phantom, "\lambda_X"] & FX_2 \arrow[d, "\lambda_{X_2}"] \\
GX_1 \arrow[r, "GX"', "|" marking] & GX_2
\end{tikzcd}$$

at loose maps $X_1 \overset{X}{\nrightarrow} X_2$ are companion commuters.

Thm: A $\overset{\text{cartesian adequate triple}}{(\mathcal{C}, L, R)}$-regular hyperdoctrine $P: \mathcal{C}^{op} \longrightarrow \overset{\text{cartesian 2-cat.}}{K}$ is the same as* a lax symmetric monoidal double pseudofunctor

$$\mathrm{Span}(\mathcal{C})^{op} \xrightarrow{P^{\bullet}} \mathbb{Q}\mathrm{t}(K)$$

with companion commuter laxators.

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* there are some minor conditions on $(\mathcal{C}, L, R)$.

Pf (key ideas) (Hyperdoctrine ⇒ double pseudofunctor):

- Extend P to $P^{\bullet}$ by acting on spans by pull-push

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[dr, "x_2"] & \\ X_1 & & X_2 \end{tikzcd} \longmapsto \begin{tikzcd} PX_1 \arrow[r, "P(x_1)", "\mid" marking] & PX \arrow[r, "\exists x_2", "\mid" marking] & PX_2 \end{tikzcd}$$

↪ see Moeller & Vasilakopoulou 2020
Shulman 2008

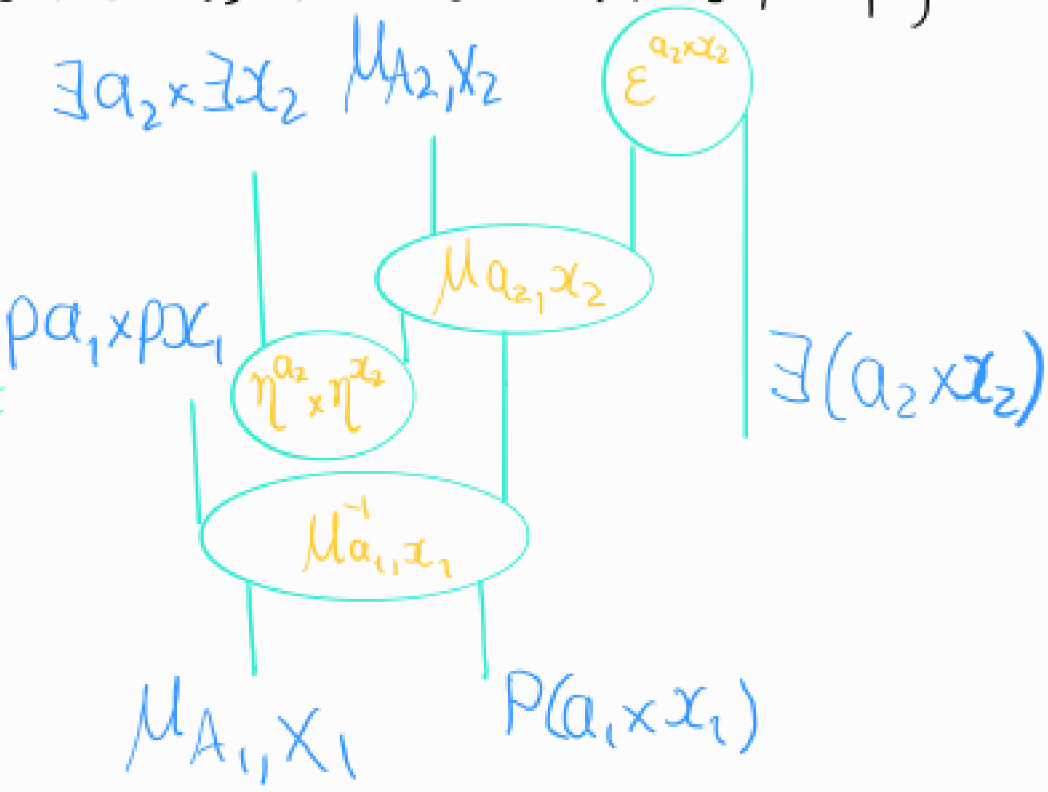
- The monoidal structures on the fibres PX induce a monoidal structure on P;

$$\mu_{X,Y} := PX \times PY \xrightarrow[P(\pi_X)\times P(\pi_Y)]{} P(X\times Y)\times P(X\times Y) \xrightarrow[\otimes_{P(X\times Y)}]{} P(X\times Y)$$

- The monoidal laxators for P extend to tight transformations that serve as laxators for $P^{\bullet}$;

e.g.:

$$\mu^{\bullet}_{A_1 \overset{A}{\nrightarrow} A_2,\, X_1 \overset{X}{\nrightarrow} X_2} :=$$



- Use the Beck-Chevalley and Frobënius cells to build an inverse to the companion squares of the $\mu^{i}_{A,X}$.

(Double functor to hyperdoctrine)

The tight component $\mathcal{C}^{op} \xrightarrow{Q_0} \mathcal{K}$ of $\mathrm{Span}(\mathcal{C})^{op} \xrightarrow{Q} Qt(\mathcal{K})$ is a regular hyperdoctrine:

- (Unitary) double functors preserve companions and conjoints;

  ⇝ gives an adjunction $Q(f!) \dashv Q(f)$ internal to $\mathcal{K}$ ⇝ Dawson, Paré, Pronk 2010

- Defining $\exists f := Q(f!)$ makes Q_0 a B.C-fibration
- The monoidal structure on Q induces one on the fibres QX:

$$QX \times QX \xrightarrow{M_{X,X}} Q(X \times X) \xrightarrow{Q(\Delta_X)} QX$$

- The companion commuters can be used to build inverses to the Frobënius maps.

Def: A regular double hyperdoctrine is a lax symmetric monoidal double pseudofunctor

symm. monoidal semantic double cat.

$$\mathbb{C}tx^{op} \xrightarrow{Q} \mathbb{D}$$

B.C double cat. of contexts

with companion commuter laxators.

Work-in-progress

- $\mathbb{D}\mathrm{bl}^{ps}_t$ admits the construction of algebras;

$$\begin{tikzcd}
\mathbb{D}\mathrm{bl}^{ps}_t \arrow[r, bend left, "i \text{ ident.}"] \arrow[r, phantom, "\perp"] & \mathrm{Mnd}(\mathbb{D}\mathrm{bl}^{ps}_t) \arrow[l, bend left, "\mathrm{IAlg}(-)"]
\end{tikzcd}$$

- $\mathrm{Span} : \mathrm{Cat}_{pb} \longrightarrow \mathbb{D}\mathrm{bl}^{ps}_t$ preserves lax limits (thus E-M objects);

thus: a lax map of monads

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "P"] \arrow[d, "I^{op}"'] & \mathrm{Pos} \arrow[d, "L"] \arrow[dl, Rightarrow, shorten=12pt, "\rho"] \\
\mathcal{C}^{op} \arrow[r, "P"'] & \mathrm{Pos}
\end{tikzcd}$$

logical enhancement monad

addition of behaviours
(I a pb-preserving comonad on $\mathcal{C}$)

induces

$$\begin{tikzcd}
\mathbb{A}\mathrm{lg}(\mathbb{S}\mathrm{pan}(I^{op}))^{op} \arrow[r, "\mathbb{A}\mathrm{lg}(\varrho)"] \arrow[d, phantom, sloped, "\cong"] & \mathbb{A}\mathrm{lg}(\mathbb{C}\mathrm{at}(L))^{op} \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{S}\mathrm{pan}(\mathrm{coalg}(I))^{op} & \mathbb{C}\mathrm{at}(\mathrm{alg}(L))^{op}
\end{tikzcd},$$

an existential double hyperdoctrine.

E.g.:

I = stream comonad

L = free temporal algebra monad

$P = \mathrm{Sub} : \mathrm{Set}^{op} \to \mathrm{Pos}$

$\varrho_X : L\,\mathrm{Sub}(X) \longrightarrow \mathrm{Sub}(X^{\mathbb{N}})$

temporal $\varphi(s) \longmapsto \{\text{streams that } \models \varphi\}$

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Obrigado!