Galois Theory of Differential & Difference Schemes

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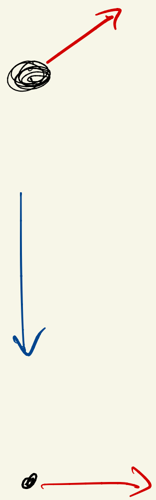
Classical Picard-Vessiot

1883 - 2023



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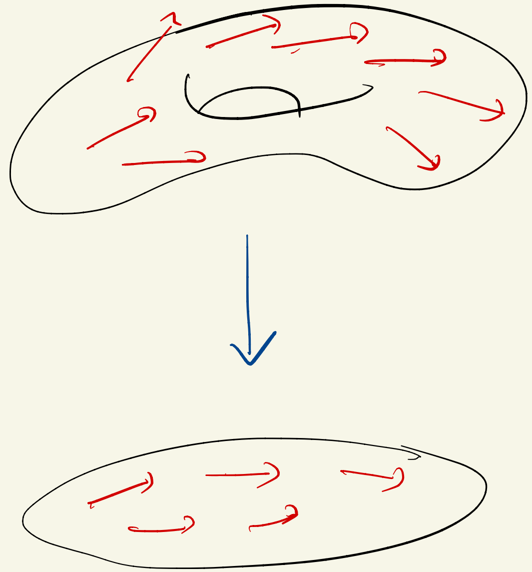


CLASSICAL PICARD-VESSIOT THEORY


$$\begin{tikzcd} (A, \delta_A) \arrow[r, hook] & (L, \delta_L) \\ & (K, \delta_K) \arrow[ul] \arrow[u] \end{tikzcd}$$

PV extension of differential fields with

- PV ring (A, δ_A)
- $k = \mathrm{Const}(K) = \mathrm{Const}(L)$.

$\rightsquigarrow$ PV Galois group $\quad G = \mathrm{Gal}^{PV}(L/K) = \mathrm{Spec}(\underbrace{\mathrm{Const}(A \otimes_K A)}_{\text{Hopf alg}/\,k})$

G: linear alg. group / k

$$\left\{ \begin{matrix} \text{intermediate differential} \\ \text{field extensions} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{closed subgroups} \\ \text{of } G \end{matrix} \right\}$$

JANELIDZE'S CATEGORICAL FORMULATION

CATEGORICAL GALOIS THEORY :

$$\begin{tikzcd} \delta\text{-}\mathrm{Ring}^{op} \arrow[d, bend right, "\mathrm{Const}"'] \arrow[d, phantom, "\dashv"] & \simeq & \delta\text{-}\mathrm{Aff} \\ \mathrm{Ring}^{op} \arrow[u, bend right, "(-,0)"'] & \simeq & \mathrm{Aff} \end{tikzcd}$$

$$\begin{tikzcd} X = \mathrm{Spec}(A) \arrow[r] \arrow[d, "f"'] & S = \mathrm{Spec}(k) \arrow[d, equal] \\ Y = \mathrm{Spec}(K) \arrow[r] & S \end{tikzcd}$$

is GALOIS,

<u>NOT</u> L/K.

Categorical Galois group(oid)

$$G = \mathrm{Gal}[f] \simeq \mathrm{Gal}^{PV}(L/K).$$

$\mathrm{Const}(X \times_Y X)$

Self-splitting :

$$X \times_Y X \simeq X \times_{(S,0)} (G,0)$$

Torsor relation

Equivalence :

$$\mathrm{Split}[f] \simeq \mathrm{Aff}^G$$

objects in $\delta\text{-}\mathrm{Aff}/_Y$ split by f

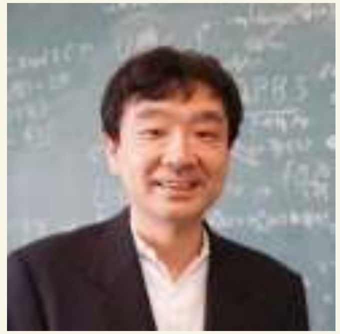
G-actions in $\mathrm{Aff}/_S$

CARBONI-JANELIDZE-MAGID CORRESPONDENCE



$$\left\{\text{SPLIT QUOTIENTS of } \begin{array}{c}X\\ \downarrow\\ Y\end{array}\right\} \overset{1:1}{\rightsquigarrow} \left\{\begin{array}{l}\text{EFFECTIVE}\\ \text{SUBGROUPS of } G\end{array}\right\}$$

$$X/H \longleftarrow\!\!\shortmid\; H \leq G$$



MASUOKA (Sept 2023): Does Categorical Galois theory recover classical PV correspondence?

Answer: NO! $H \leq G$ is effective in Aff iff G/H affine scheme.

In general G/H can be QUASI-PROJECTIVE.

$\rightsquigarrow$ We must work with GENERAL DIFFERENTIAL SCHEMES.

$$\begin{tikzcd} \delta\text{-Sch} \arrow[d, dashed, bend right=40, "?"'] \arrow[d, phantom, "\dashv"] \\ \text{Sch} \arrow[u, bend right=40, "C=(-,0)"'] \end{tikzcd}$$

DIFFERENTIAL SCHEMES

S - scheme

S - differential scheme : $(X, (\mathcal{O}_X, \delta_X))$, where $(X, \mathcal{O}_X) \in Sch/_S$

$\delta_X \in Der_S(\mathcal{O}_X, \mathcal{O}_X)$.

a vector field on X.

$\rightsquigarrow$ category

$\delta\text{-}Sch_S$

Functor

$$C : Sch/_S \longrightarrow \delta\text{-}Sch_S$$
$$(X, \mathcal{O}_X) \longmapsto (X, (\mathcal{O}_X, 0)).$$

CATEGORICAL SCHEME OF LEAVES

[inspired by BARDAVID]

If $\eta : (X, \delta_X) \longrightarrow (Q, 0) = C(Q)$ is universal from X to C, i.e.

$$\begin{tikzcd}
(X, \delta_X) \arrow[r, "\eta"] \arrow[dr, bend left, "\forall \eta'"'] & C(Q) \arrow[d, dashed, "\exists !"] \\
& C(Q')
\end{tikzcd}$$

then $\quad Q = \pi_0(X, \delta_X)$

- rarely exists
- π_0 PARTIAL left adjoint to C
- in general

$$\pi_0(\mathrm{Spec}(A, \delta_A)) \neq \mathrm{Spec}(\mathrm{Const}(A, \delta_A))$$

AN INTERLUDE: INTERNAL PRECATEGORIES

$\mathcal{A}$ – a category.

a <u>precategory</u> $\mathbb{C} \in \mathrm{PreCat}(\mathcal{A})$ is a diagram in $\mathcal{A}$

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "r_0"] \arrow[r, "m" description] \arrow[r, shift right=2, "r_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "n" description] \end{tikzcd}$$

s.t.

$$d_0 r_1 = d_1 r_0 \qquad d_0 n = id_{C_0}$$

$$d_0 m = d_0 r_0 \qquad d_1 n = id_{C_0}$$

$$d_1 m = d_1 r_1$$

$\mathbb{C}$ is a <u>category</u> if $C_2 \simeq C_1 \times_{C_0} C_1$

Example :

Kernel pair groupoid of

$$f : X \to Y \quad \text{in } \mathcal{A}$$

$\mathbb{G}_f$:

$$X \times_Y X \times_Y X \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; X \times_Y X \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X \qquad \in \mathrm{Cat}(\mathcal{A}) .$$

INDEXED DATA

PSEUDOFUNCTOR $\mathcal{P} : \mathcal{A}^{op} \to CAT$

- for $X \in \mathcal{A}$, $\mathcal{P}(X)$ is a category
- for $X \xrightarrow{f} Y$ in $\mathcal{A}$, $f^* = \mathcal{P}(f) : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ functor

\+ coherence

EXAMPLES:

① $\mathcal{A} = Ring^{op}$

$\mathcal{P}(R) = R\text{-Mod}$

② $\mathcal{A} = Sch$

$\mathcal{P}(X) = QCoh(X)$

③ $\mathcal{A} = Sch$

$\mathcal{P}(X) = \left\{ \begin{array}{c} Q \\ q \downarrow \\ X \end{array} : q \text{ quasiprojective} \right\}$

LAX PRECATEGORY ACTIONS : ABSTRACT DESCENT DATA

$\mathbb{C} \in \mathrm{PreCat}(\mathcal{A})$

$\mathcal{P} : \mathcal{A}^{op} \longrightarrow \mathrm{CAT}$ pseudofunctor

Category of (lax) $\mathbb{C}$-actions in $\mathcal{P}$:

$P = (P_0, \varphi) \in \mathcal{P}^{\mathbb{C}}_{lax}$, where

- P_0 in $\mathcal{P}(C_0)$
- $d_1^* P_0 \xrightarrow{\varphi} d_0^* P_0$ in $\mathcal{P}(C_1)$

\+ cocycle condition in $\mathcal{P}(C_2)$

$$
\begin{tikzcd}
n^*d_1^*P_0 \arrow[rr, "n^*\varphi"] & & n^*d_0^*P_0 \\
& P_0 \arrow[ul, "\sim"'] \arrow[ur, "\sim"] &
\end{tikzcd}
$$

$\leftarrow$ in $\mathcal{P}(C_0)$

$[n^*\varphi \simeq \mathrm{id}]$

$$
\begin{tikzcd}
& m^*d_1^*P_0 \arrow[rr, "m^*\varphi"] & & m^*d_0^*P_0 \arrow[dr, "\sim"] & \\
r_1^*d_1^*P_0 \arrow[ur, "\sim"] \arrow[dr, "r_1^*\varphi"'] & & \text{COCYCLE CONDITION} & & r_0^*d_0^*P_0 \\
& r_1^*d_0^*P_0 \arrow[rr, "\sim"] & & r_0^*d_1^*P_0 \arrow[ur, "r_0^*\varphi"'] &
\end{tikzcd}
$$

$\leftarrow$ in $\mathcal{P}(C_2)$

$[r_0^*\varphi \circ r_1^*\varphi \simeq m^*\varphi]$

A FIBRATIONAL VIEW OF PRECATEGORY ACTIONS

$$\mathcal{P} : \mathcal{A}^{op} \longrightarrow CAT \qquad \rightsquigarrow \qquad \begin{array}{c} \tilde{\mathcal{P}} = \int \mathcal{P} \\ \downarrow \\ \mathcal{A} \end{array}$$

$$\mathbb{C} \in PreCat(\mathcal{A})$$

$$P \in \mathcal{P}^{\mathbb{C}}_{lax} \qquad \leftrightsquigarrow$$

$$\begin{tikzcd}
\mathbb{P} \cdots & P_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & P_1 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & P_0 \in PreCat(\tilde{\mathcal{P}}) \\
\mathbb{C} \cdots \arrow[u, dotted, no head] & C_2 \arrow[u, dotted, no head] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & C_1 \arrow[u, dotted, no head] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & C_0 \arrow[u, dotted, no head]
\end{tikzcd}$$

" $\longrightarrow$ " CARTESIAN in $\tilde{\mathcal{P}}$.

$\longrightarrow$ ARBITRARY

PRECATEGORICAL DESCENT

$f : \mathbb{C} \to \mathbb{D}$ morphism in $\mathrm{Precat}(\mathcal{A})$

$\mathcal{P} : \mathcal{A}^{op} \to \mathrm{CAT}$ pseudofunctor.

- f is descent for $\mathcal{P}$ if $f^* : \mathcal{P}^{\mathbb{D}} \to \mathcal{P}^{\mathbb{C}}$ is f.f.
- effective descent —— " —— equivalence

CLASSICAL DESCENT

$$\begin{array}{c} X \\ f\big\downarrow \\ Y \end{array} \text{ in } \mathcal{A} \rightsquigarrow \begin{array}{c} \mathcal{P}(X) \\ \big\uparrow f^* \\ \mathcal{P}(Y) \end{array}.$$

Question: which $P \in \mathcal{P}(X)$ are of form $P \simeq f^*Q$ for some $Q \in \mathcal{P}(Y)$?

Recall $G_f \in \mathrm{Cat}(\mathcal{A})$:

$$X \times_Y X \times_Y X \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; X \times_Y X \;\substack{\xrightarrow{d_0}\\\longleftarrow\\\xrightarrow[d_1]{}}\; X$$

Such a $P \in \mathcal{P}(X)$ has a glueing morphism $d_1^*P \xrightarrow{\varphi} d_0^*P$ in $\mathcal{P}(X \times_Y X)$

+ a cocycle condition in $\mathcal{P}(X \times_Y X \times_Y X)$.

$\rightsquigarrow$ category of descent data

$$DD_{\mathcal{P}}(f)$$

CLASSICAL VS PRECATEGORICAL DESCENT

Refined question: when is

$$f^*: \mathcal{P}(Y) \longrightarrow DD_{\mathcal{P}}(f)$$

- an equivalence (f effective descent)
- fully faithful (f descent)
- faithful (f pre-descent) ?

Note:

$$DD_{\mathcal{P}}(f) \simeq \mathcal{P}^{G_f}$$

$\leadsto$ f is classically descent / effective descent

iff the $\mathrm{Precat}(\mathcal{A})$-morphism

$$G_f \longrightarrow G_{id_Y} = \Delta(Y) \quad \text{is.}$$

DIFFERENTIAL SCHEMES AS PRECATEGORY ACTIONS

PRECATEGORY in $Sch_{/S}$

$$\mathbb{D}(S): \quad S[\varepsilon_1,\varepsilon_2]/(\varepsilon_1^2,\varepsilon_1\varepsilon_2,\varepsilon_2^2) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; S[\varepsilon]/(\varepsilon^2) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; S$$

Key observation :

$$\delta\text{-}Sch_S \simeq \left(Sch_{/S}\right)^{\mathbb{D}(S)}$$

$$(X,\delta_X) \rightsquigarrow \mathbb{X} = \left(X_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_0 \right)$$

CATEGORICAL SCHEME of LEAVES $\rightsquigarrow$ CONNECTED COMPONENTS :

$$\pi_0(X,\delta_X) := \pi_0(\mathbb{X}) = \operatorname{coeq}\left(X_1 \rightrightarrows X_0\right)$$

when it exists in Sch.

DIFFERENTIAL DESCENT

$$\mathcal{P} : (\mathrm{Sch}/_S)^{op} \longrightarrow \mathrm{CAT} \quad \rightsquigarrow \quad \delta\text{-}\mathcal{P} : (\delta\text{-}\mathrm{Sch}_S)^{op} \longrightarrow \mathrm{CAT}$$

$$(X, \delta_X) \longmapsto \mathcal{P}^{\mathbb{X}}$$

precategory $\mathbb{X}$
actions in $\mathcal{P}$.

$$\begin{array}{c} (X,\delta_X) \\ f \big\downarrow \\ (Y,\delta_Y) \end{array} \quad \rightsquigarrow \quad \begin{tikzcd} \mathbb{X} \arrow[d] & \cdots & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "f_2"'] & X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "f_1"] & X_0 \arrow[l] \arrow[d, "f_0"] \\ \mathbb{Y} & \cdots & Y_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & Y_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & Y_0 \arrow[l] \end{tikzcd}$$

<u>Th</u> f is effective descent for δ-$\mathcal{P}$ if f_0 effective descent for $\mathcal{P}$,
f_1 descent for $\mathcal{P}$,
and f_2 pre-descent for $\mathcal{P}$.

SIMPLICITY & PRECATEGORICAL DESCENT

<u>Def</u> (X, δ_X) is SIMPLE wrt $\mathcal{P} : (Sch_{/S})^{op} \longrightarrow CAT$

if $\pi_0(X, \delta_X) = \pi_0(\mathbb{X})$ exists and is universal for $\mathcal{P}$.

$$
\rightsquigarrow \quad \eta_X : \begin{tikzcd}
\mathbb{X} \arrow[d] & \cdots & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & X_1 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d] & X_0 \arrow[d] \\
\Delta(\pi_0(X)) & \cdots & \pi_0(X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \pi_0(X) \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \pi_0(X)
\end{tikzcd}
$$

is of PRECATEGORICAL DESCENT,

i.e.

$$
\begin{array}{ccc}
\mathcal{P}^{\Delta(\pi_0(X))} & & \mathcal{P}^{\mathbb{X}} \\
\parallel & & \parallel \\
C_X : \mathcal{P}(\pi_0(X)) & \longrightarrow & \delta\text{-}\mathcal{P}(X, \delta_X) \\
Q & \longmapsto & \eta_X^* Q
\end{array}
$$

is fully faithful.

[think $\begin{array}{c} Q \\ \downarrow \\ \pi_0(X) \end{array} \longmapsto (X, \delta_X) \times_{(\pi_0(X), 0)} (Q, 0)$ is f.f., i.e. $X_1 \rightrightarrows X_0 \longrightarrow \pi_0(X)$ univ. coeq.]

INDEXED CATEGORICAL GALOIS TH. [Borceux - Janelidze]

- $$\begin{tikzcd} \mathcal{A} \arrow[rr, "L"] \arrow[d, "S"'] \arrow[drr, phantom, "\overset{\alpha}{\Nearrow}"] & & \mathrm{CAT} \\ \mathcal{X} \arrow[urr, "K"'] & & \end{tikzcd}$$

  functor S, pseudofunctor L

- $X \xrightarrow{f} Y \in \mathcal{A}$ s.t. α_X, $\alpha_{X \times_Y X}$, $\alpha_{X \times_Y X \times_Y X}$ f.f.

- f effective descent wrt L

$$\Longrightarrow \qquad \mathrm{Split}_\alpha(f) \simeq K^{S \circ \mathbb{G}_f}$$

INDEXED FRAMEWORK FOR DIFFERENTIAL GALOIS TH.

that admit π_0

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d, bend right, "\pi_0"'] \arrow[d, phantom, "\dashv"] & \delta\text{-}Sch_S \\ \mathcal{X} \arrow[u, bend right, "C"'] & \end{tikzcd}$$

$$\mathcal{X} = Sch/_S$$

- $\mathcal{P} : \mathcal{X}^{op} \longrightarrow CAT$

$\rightsquigarrow \delta\text{-}\mathcal{P} : \mathcal{A}^{op} \longrightarrow CAT$

- $C : \mathcal{P} \circ \pi_0 \Rightarrow \delta\text{-}\mathcal{P}$ pseudonatural

$C_Z : \mathcal{P}(\pi_0(Z)) \longrightarrow \delta\text{-}\mathcal{P}(Z)$ as before.

(PRE-) PICARD-VESSIOT MORPHISMS

<u>Def</u>

$f : (X, \delta_X) \longrightarrow (Y, \delta_Y) \in \mathcal{A}$ is

pre-PV for $\mathcal{P}$, if :

(1) f is effective descent for δ-$\mathcal{P}$

(2) X, $X \underset{Y}{\times} X$, $X \underset{Y}{\times} X \underset{Y}{\times} X$ are simple for $\mathcal{P}$,

<u>Def</u> (for suitable $\mathcal{P}$)

f is PV if:

(1) — " —

(2) X simple & $f \in \mathrm{Split}[f]$.

self-splitting

GALOIS PRECATEGORY / GROUPOID

- f PV $\Rightarrow$ f pre-PV

- f pre-PV $\rightsquigarrow$ $\mathbb{G}_f = \left(X \times_Y X \times_Y X \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; X \times_Y X \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; X \right) \in \mathrm{Cat}(\mathcal{A})$

  $\rightsquigarrow$ $\mathrm{Gal}[f] = \pi_0(\mathbb{G}_f) \in \mathrm{PreCat}(\mathcal{X})$ GALOIS PRECATEGORY.

- f PV $\Rightarrow$ $\mathrm{Gal}[f] \in \mathrm{Cat}(\mathcal{X})$ GROUPOID.

GALOIS THEORY OF DIFFERENTIAL SCHEMES

<u>Th</u> • f pre-PV $\Rightarrow$ equivalence

$$\mathrm{Split}_C[f] \simeq \mathcal{P}^{\mathrm{Gal}[f]}.$$

• f PV $\Rightarrow$ RHS: groupoid actions.

<u>Pf</u> f pre-PV $\Rightarrow$ (1) f is effective descent

(2) $X, X \times_Y X, X \times_Y X \times_Y X$ simple $\Rightarrow C_X, C_{X \times_Y X}, C_{X \times_Y X \times_Y X}$ f.f.

$\Rightarrow$ Janelidze's indexed Galois Th. applies.

APPLICATIONS: QUASI-PROJECTIVE THEORY

- (K, δ_K) diff. field char 0, $k = \mathrm{Const}(K)$, $S = \mathrm{Spec}(k)$,
- $(X, \delta_X) \xrightarrow{f} (Y, \delta_Y) = \mathrm{Spec}(K, \delta_K)$ quasi-projective integral, only leaf is generic pt.
- f is self-split.

$\Rightarrow$ f is PV, $G = \mathrm{Gal}[f]$ is an S-group scheme, correspondence:

Split quotients

$$\begin{tikzcd} & X \arrow[dl, "\text{fpqc}"'] \arrow[dd, "f"] \\ P \arrow[dr, "p", "\text{quasi-proj.}"'] & \\ & Y \end{tikzcd}$$

$\longleftrightarrow$

Closed subgroups

$G' \leq G$

$X/G' \longleftarrow\!| \; G'$

$p \longmapsto \mathrm{Gal}[X \to P]$

$\rightsquigarrow$ unifies linear PV theory and STRONGLY NORMAL th. of KOLCHIN.

EXAMPLE : RELATIVE ELLIPTIC CURVE

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

$S[z]$

Weierstrass family of elliptic curves

$y^2 = x^3 + ux + v$

$S = \mathrm{Spec}\left(\mathbb{Q}\left[u, v, \frac{1}{4u^3 + 27v^2}\right]\right)$

$\Omega_{X/S} = \langle \omega_X, dz \rangle$

$\delta_X : dz \mapsto 1$

$\omega_X \mapsto z$

"$\ell d(x) = z$"

$\delta_Y : dz \mapsto 1$

$\Omega_{E/S} = \langle \omega_0 \rangle$

invariant differential $\frac{dx}{y}$.

f is PV $\rightsquigarrow$ $\mathrm{Gal}[f] \simeq (E \rightrightarrows S)$

SPECIALISATION: for $s \in S(L)$,

$$\mathrm{Gal}[f_s] \simeq \mathrm{Gal}[f]_s \simeq E_s$$

EXAMPLE : AIRY EQUATION

$$\begin{array}{l} X = \mathbb{A}^1 \times GL_2 = \operatorname{Spec}\left(k\left[x, u, \frac{1}{\det u}\right]\right) \\ \quad \big\downarrow f \\ Y = \mathbb{A}^1 \end{array}$$

with vector field

$$\frac{\partial}{\partial x} + u_{21}\frac{\partial}{\partial u_{11}} + u_{22}\frac{\partial}{\partial u_{12}} + x u_{11}\frac{\partial}{\partial u_{21}} + x u_{12}\frac{\partial}{\partial u_{22}} \;; \quad \omega \longmapsto \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$$

f is PV $\rightsquigarrow$ Galois groupoid

$$\begin{array}{ccc} Gal[f] : \pi_0\left(X \times_Y X\right) & \rightrightarrows & \pi_0(X) \\ \wr\wr & & \wr\wr \\ \left(GL_2 \times GL_2\right)/SL_2 & \rightrightarrows & GL_2/SL_2 \simeq \mathbb{G}_m \end{array}$$

isomorphisms between PV extensions

Analogy : Deligne's fundamental groupoid (Cat. Tannakiennes)

FORTHCOMING WORK

Differential Galois Theory = (precategorical) Descent + Categorical Galois Theory

$\rightsquigarrow$ difference PV-style Galois theory [uses lax actions]

$\rightsquigarrow$ common generalizations:
- σ-δ theory [Di Vizio-Hardouin-Wibmer]
- δ-δ theory [Cassidy-Singer]
- partial case
- Hasse-Schmidt derivatives
- Lie algebra formal actions.