Simplicial Resolutions in Model Categories and a Perspective toward Multisimplicial Constructions in Higher Categories

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Model Categories

Idea.

- Introduced by Quillen to formalize homotopy theory in a categorical setting.
- Examples include **Top**, **SSet**, and chain complexes.
- Provide the right context for constructing simplicial resolutions.

Formal Definition

Definition

A **model structure** on a category $\mathcal{C}$ consists of:

- three classes of morphisms: **weak equivalences**, **fibrations**, **cofibrations**;
- two functorial factorizations, ensuring every map can be written in two canonical ways.

These satisfy the axioms of:

- **Retracts:** closed under retracts;
- **2-of-3:** if $f, g, g \circ f$ are composable, any two weak equivalences imply the third;
- **Lifting:** cofibrations lift against acyclic fibrations and vice versa;
- **Factorization:** every map factors as cofibration–fibration or as acyclic cofibration–acyclic fibration.

Why Model Categories?

- Provide a categorical presentation of homotopy theories.
- Allow construction of derived functors and homotopy (co)limits.
- Natural framework for building **simplicial and multisimplicial resolutions**.

Why Simplicial Resolutions?

- Classical resolutions use projectives, but not all categories are additive (e.g., groups).
- Simplicial resolutions extend projective resolutions via simplicial kernels and resolving subcategories.
- Useful in categories like Grp, Top, SSet, and model categories.

Simplicial Kernels

Definition

Let $\mathcal{C}$ be a category and $X, Y \in \mathrm{Ob}\,\mathcal{C}$. Suppose $n > 0$ and a tuple of morphisms $(f_i : X \to Y)_{i\in[0,n]}$.
A *simplicial kernel* (or n-equalizer) of (f_i) is a pair

$$\big(K, (k_i : K \to X)_{i\in[0,n+1]}\big)$$

such that:

(i) $k_j \circ f_i = k_i \circ f_{j-1}$ for all $0 \leq i < j \leq n+1$;

(ii) For any object Z with morphisms $(h_i : Z \to X)_{i\in[0,n+1]}$ satisfying $h_j \circ f_i = h_i \circ f_{j-1}$ for all $i < j$, there exists a unique morphism $\mu : Z \to K$ such that $\mu \circ k_i = h_i$ for all i.

- Encodes higher coherence among face maps.
- Constructed via a **reduced limit** over a specific diagram.

Resolving Subcategories

Definition

Let $\mathcal{C}$ be a category and $\mathcal{P} \subset \mathcal{C}$ a full subcategory. A morphism $\varphi : X \to Y$ is called *$\mathcal{P}$-epic* if for every $P \in \mathrm{Ob}\ \mathcal{P}$ and morphism $\alpha : P \to Y$, there exists a morphism $\beta : P \to X$ such that:

$$\varphi \circ \beta = \alpha.$$

Definition

Let $\mathcal{C}$ be a category. A full subcategory $\mathcal{P} \subset \mathcal{C}$ is called a *resolving subcategory* if for every object $X \in \mathrm{Ob}\ \mathcal{C}$, there exists a $P \in \mathcal{P}$ and a $\mathcal{P}$-epimorphism $\varphi : P \to X$.

Step 0: Initial Object and $\mathcal{P}$-Epimorphism

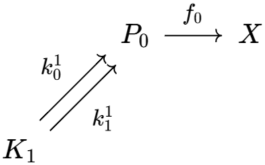
Let $\mathcal{C}$ be a category with finite limits, $\mathcal{P}$ a resolving subcategory, and $X \in \mathcal{C}$.

1. Choose $P_0 \in \mathcal{P}$ and a $\mathcal{P}$-epimorphism $f_0 : P_0 \to X$

$$P_0 \xrightarrow{f_0} X$$

Step 1: First Simplicial Kernel

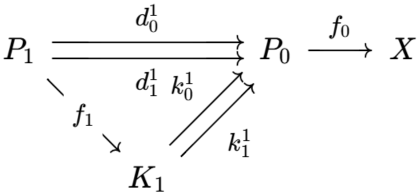
2. Construct simplicial kernel $(K_1, (k_0, k_1))$ of $f_0 : P_0 \to X$



- K_1 encodes relations: $f_0 \circ k_0 = f_0 \circ k_1$

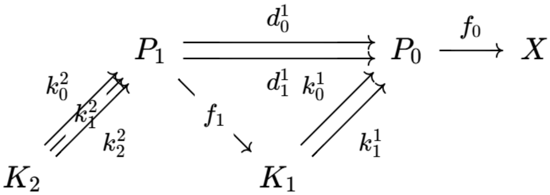
Step 2: First Level Resolution

3. Choose $P_1 \in \mathcal{P}$ and $\mathcal{P}$-epimorphism $f_1 : P_1 \to K_1$
4. Define face maps: $d_0^1 := f_1 \circ k_0$, $d_1^1 := f_1 \circ k_1$



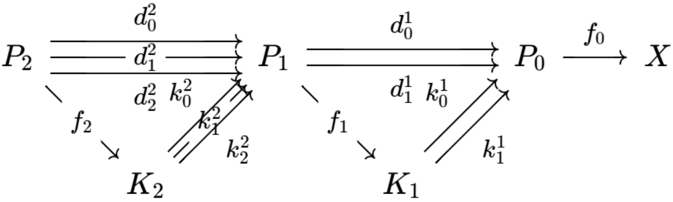
Step 3: Second Simplicial Kernel

5. Construct simplicial kernel $(K_2, (k_0^2, k_1^2, k_2^2))$ of (d_0^1, d_1^1)



Step 4: Second Level Resolution

6. Choose $P_2 \in \mathcal{P}$ and $\mathcal{P}$-epimorphism $f_2 : P_2 \to K_2$
7. Define face maps: $d_i^2 := f_2 \circ k_i^2$ for $i = 0, 1, 2$



Step 5: Inductive Construction

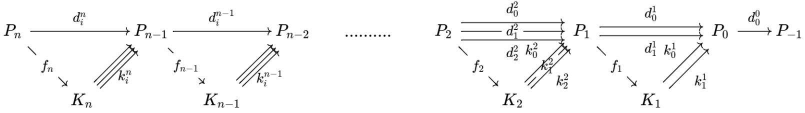
For $n \geq 2$:

1. Assume constructed $(P_{n-1} \xrightarrow{d_i^{n-1}} P_{n-2})_{i=0}^{n-1}$
2. Construct simplicial kernel $(K_n, (k_i^n)_{i=0}^n)$ of $(d_i^{n-1})_{i=0}^{n-1}$
3. Choose $P_n \in \mathcal{P}$ and $\mathcal{P}$-epimorphism $f_n : P_n \to K_n$
4. Define face maps: $d_i^n := f_n \circ k_i^n$ for $i = 0, \ldots, n$

Resulting Semi-simplicial Resolution

The construction yields:

- Semi-simplicial resolution: $((P_n)_{n\geq 0}, (d_i^n)_{0\leq i\leq n}^{n\geq 1})$
- Augmented Semi-simplicial resolution: $((P_n)_{n\geq -1}, (d_i^n)_{0\leq i\leq n}^{n\geq 0})$ where $P_{-1} := X$ and $d_0^0 := f_0$



From Semi-simplicial to Simplicial

- The kernel construction gives a **semisimplicial object** R with $R[n] = P_n$.
- Problem: it lacks **degeneracies**.
- Solution: define a functor F_C that freely adds degeneracies, left adjoint to the forgetful functor V_C.

Construction (idea)

- For each n, set
$$(F_C X)_n = \coprod_{[n] \twoheadrightarrow [k]} X_k.$$
- Each surjection $[n] \twoheadrightarrow [k]$ represents a degeneracy.
- Result: $F_C X$ is a **simplicial object** and $F_C \dashv V_C$.

Application to Resolutions

- If $P \subset C$ is a resolving subcategory closed under finite coproducts, then R semisimplicial $\mapsto$ $F_C R$ simplicial.
- Result: $F_C R$ is the desired **simplicial resolution** of X.

Introduction to Higher Categories

- We start from classical 1-categories: they record objects and arrows, but homotopy forces us to see arrows *between* arrows.
- So we move to higher morphisms: 2-morphisms, 3-morphisms, and so on.
- In a weak n-category, composition and units are not strict equalities; they hold up to higher equivalence, witnessed by higher cells.
- An ∞-category pushes this all the way: coherences exist at every level. A key model is a *quasi-category*, a simplicial set where all inner horns can be filled.
- Why this matters for us: simplicial resolutions already capture 1-dimensional homotopies; higher categories let us organize the higher coherences—the relations among relations that appear along a resolution.

Coherences in Higher Categories

- In higher settings, associativity and unit laws are controlled by coherence data—think Mac Lane's pentagon and triangle.
- Basic simplicial identities already encode low-level coherence via faces and degeneracies.
- But beyond that, we need extra structure to keep track of all higher compatibilities.
- This is the heart of *strict vs. weak*: strict equalities rarely survive homotopy-invariant settings; weak structures replace equality by equivalence with coherent fillers.
- For resolutions, this captures "syzygies of syzygies"—higher relations that generalize projective resolutions outside additive categories.

Nerves and Simplicial Models of Higher Categories

- The nerve functor sends a category C to a simplicial set $N(C)$ whose k-simplices are chains of k composable morphisms.
- Composition becomes a horn-filling problem: filling an inner horn corresponds to choosing a composite.
- Quasi-categories require exactly those inner horn fillers, modeling $(\infty, 1)$-categories where composition and coherence are encoded simplicially.
- The Segal perspective formalizes "composition up to homotopy" via Segal maps; multi-Segal versions handle several directions at once.
- For resolutions, working inside these models means the coherences come for free from the simplicial encoding.

Ideas for Multisimplicial Resolutions — I (Inductive Multikernels)

- Goal: generalize a simplicial resolution $P_\bullet \to X$ to a multi-indexed object $P_{\bullet,\bullet} \to X$ (or more generally a $(\Delta^{\mathrm{op}})^n$-diagram).
- Horizontally, build $R_\bullet \to X$ using simplicial kernels and $\mathcal{P}$-epimorphisms, exactly as before.
- Vertically, for each k, apply the same kernel construction to R_k to get a vertical simplicial direction $K^{\mathrm{vert}}_{\bullet,k}$.
- Ask that the resolving subcategory $\mathcal{P}$ is closed under multikernels, so these iterated limits stay inside $\mathcal{P}$.
- Finally, ensure that the diagonal $\mathrm{diag}(P_{\bullet,\bullet}) \to X$ is a weak equivalence in the ambient model category.

Ideas for Multisimplicial Resolutions — II (Free Multidegeneracies & Segal)

- To add degeneracies independently in each direction, use the free multidegeneracy functor $F_C^{(n)} : \mathrm{Semi}^n C \to s^n C$, left adjoint to the forgetful $V_C^{(n)}$.
- Workflow: first build a semi-multisimplicial object via iterated kernels; then apply $F_C^{(n)}$ to obtain a full n-simplicial resolution.
- Impose Segal-type conditions in each direction: the Segal map from $P_{k,\dots}$ to the iterated fiber product of $P_{1,\dots}$ over $P_{0,\dots}$ should be a weak equivalence,

$$P_{k,\dots} \longrightarrow \underbrace{P_{1,\dots} \times_{P_{\mathbf{0},\dots}} \cdots \times_{P_{\mathbf{0},\dots}} P_{1,\dots}}_{k \text{ times}}.$$

- This enforces compositional coherence across directions.
- Outlook: prove homotopy invariance and uniqueness up to homotopy (e.g., via complete Segal machinery) and validate on low-dimensional test cases.

Conclusion

- Model categories provide the right context for building simplicial resolutions.
- Simplicial kernels and resolving subcategories yield semi-simplicial resolutions; free functors add degeneracies to obtain simplicial ones.
- Higher-categorical viewpoints clarify how to extend to **multisimplicial resolutions** that track multi-level coherences.
- Practical paths: (i) inductive multikernel constructions, (ii) free multidegeneracies with Segal-type conditions, aiming at homotopy-coherent multiresolutions.

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Thank you for your attention!