(Weakly) protomodular objects in unital categories

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CIDMA, UAlg, Portugal

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- **Answer**: No (we give an example)

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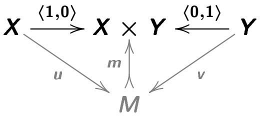

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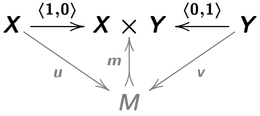

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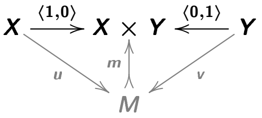

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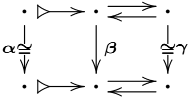

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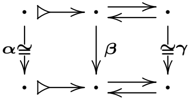

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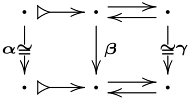

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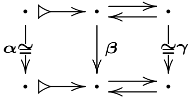

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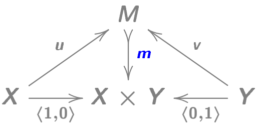

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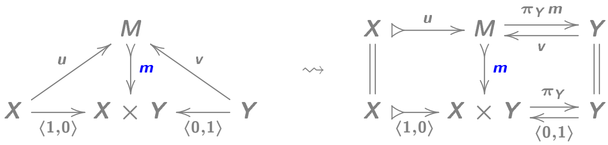

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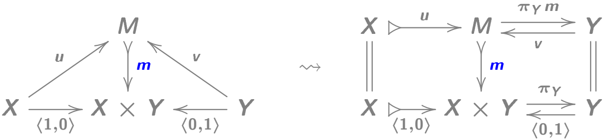

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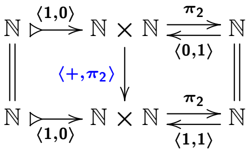

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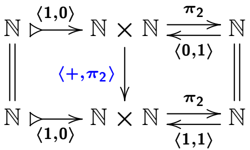

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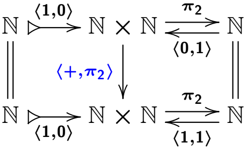

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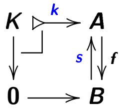

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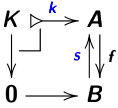

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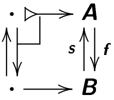

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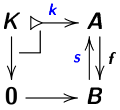

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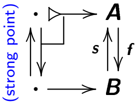

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  Ex: The category $\mathbf{Set}_*$ of pointed sets is 0+lex and **not unital**: any singleton $\{x\}$ is weakly protomodular but not protomodular

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  Ex: A group is a protomodular object in **Grp** (**Grp** is a protomodular category)

  A group is a protomodular object in the category of unital magmas only if it is $\mathbf{0}$

(Weakly) protomodular objects - examples

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Main example

[G-MMRVdL, *A comparison between weakly protomodular and protomodular objects in unital categories*, arXiv:2409.19076]

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Main example - cont

- **Thm**: $X \in \mathbb{L}\mathbf{PM}$ is weakly protomodular iff $\forall x \in X, \exists x_1, \cdots, x_n \in X$:

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Main example - cont

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