XV Portuguese Category Seminar
Pushforward and ternary semidirect products in semi-abelian categories

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Pushforwards of short exact sequences

Given a short exact sequence

$$K \xrightarrow{k} X \xrightarrow{q} Q$$

in an abelian category, and a morphism $\varphi\colon K \to L$, taking the pushout of φ and k gives a new short exact sequence

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{q} & Q \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ L & \xrightarrow[l]{} & Y & \xrightarrow[q']{} & Q. \end{array}$$

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However, this generally does not hold in non-abelian categories (such as **Grp**).

Split extensions in semi-abelian categories

If $\mathcal{C}$ is semi-abelian, then there is an equivalence between split short exact sequences and actions.
The category of B-actions is the category of $B\flat\_$-algebras, where $B\flat X$ is determined by the short exact sequence

$$B\flat X \xrightarrow{\kappa_{B,X}} B + X \underset{i_B}{\overset{[1,0]}{\rightleftarrows}} B.$$

Thus if

$$K \xrightarrow{k} X \underset{s}{\overset{p}{\rightleftarrows}} B$$

is a split exact sequence and $\varphi\colon K \to L$ is a morphism, φ can extended to a morphism of short exact sequences iff it is a morphism of B-actions.

In semi-abelian categories

Proposition (Cigoli, Mantovani, Metere)

Given a short exact sequence $K \xrightarrow{k} X \xrightarrow{q} Q$ *in a semi-abelian category* $\mathcal{C}$*, an action* $\xi \colon X \flat L \to L$ *and an* X*-equivariant morphism* $K \xrightarrow{\varphi} L$*, there exists a morphism of short exact sequences*

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{q} & Q \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ L & \xrightarrow[l]{} & Y & \xrightarrow[q']{} & Q \end{array}$$

if and only if $(\varphi \rtimes X)^* \chi = [k, 1\rangle^* \xi \colon (K \rtimes X) \flat L \to L$.

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if and only if $(\varphi \rtimes X)^*\chi = [k,1\rangle^*\xi \colon (K \rtimes X)\flat L \to L$.

But (when) is it enough to ask instead that $\varphi^*\chi_L = k^*\xi$?

We define the category $\mathbf{SES}(\mathcal{C})$ whose objects are short exact sequences and morphisms are triple

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{q} & Q \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ L & \xrightarrow[l]{} & Y & \xrightarrow[r]{} & R. \end{array}$$

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For a fixed short exact sequence

$$(S) = \ K \xrightarrow{k} X \xrightarrow{q} Q,$$

pushforwards of S can be seen as the objects of the coslice category $S \backslash \mathbf{SES}(\mathcal{C})$.

A couple facts about $\mathbf{SES}(\mathcal{C})$

Fact

$\mathbf{SES}(\mathcal{C})$ *is equivalent to the category of normal monomorphisms or epimorphisms in* $\mathbf{Arr}(\mathcal{C})$*. In particular, if* $\mathcal{C}$ *has finite limits and colimits then so does* $\mathbf{SES}(\mathcal{C})$*.*

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Fact

There are two functors $\mathcal{C} \to \mathbf{SES}(\mathcal{C})$*, sending an object* X *to the sequences*

$$I^k(X) = \; X = X \longrightarrow 0 \qquad I^c(X) \; 0 \longrightarrow X = X$$

and these functors form a chain of adjunction with the three functors $\mathbf{SES}(\mathcal{C}) \to \mathcal{C}$ *:*

$$C \dashv I^c \dashv M \dashv I^k \dashv K.$$

The composite functor

$$S\backslash\mathbf{SES}(\mathcal{C}) \longrightarrow \mathbf{SES}(\mathcal{C}) \xrightarrow{K} \mathcal{C}$$

admits a left adjoint, which sends L to the coproduct of S and $I^k(L)$:

$$\begin{array}{ccccc} K & \xrightarrow{\quad k \quad} & X & \xrightarrow{\quad q \quad} & Q \\ {\scriptstyle \bar{i}}\downarrow & & \downarrow{\scriptstyle i_X} & & \| \\ T_S(L) & \xrightarrow[\lambda]{} & X+L & \xrightarrow[{[q,0]}]{} & Q. \end{array}$$

In particular, $L \mapsto T_S(L)$ defines a monad on $\mathcal{C}$.

Proposition

The category of T_S-algebra is a coreflective subcategory of $S\backslash\mathbf{SES}(\mathcal{C})$, corresponding to short exact sequences with isomorphic cokernels.

Two special cases

For $(S) = I^c(X)$, we have $T_S(L) = X \flat L$, and the algebras for this monad are just actions of X.

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In general, T_S-algebras are formed by combining these two cases.

The decomposition of the monad

We have a diagram

$$\begin{array}{ccccc}
X\flat L & \xrightarrow{\overline{\kappa}} & T_S(L) & \underset{\overline{i}}{\overset{\overline{p}}{\rightleftarrows}} & K \\
\| & & \downarrow{\scriptstyle\lambda} & \lrcorner & \downarrow{\scriptstyle k} \\
X\flat L & \xrightarrow{\kappa_{X,L}} & X+L & \underset{i_X}{\overset{[1,0]}{\rightleftarrows}} & X \\
\downarrow & & \downarrow{\scriptstyle [q,0]} & & \downarrow{\scriptstyle q} \\
0 & \longrightarrow & Q & = & Q
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which shows that $T_S(L) \simeq (X\flat L) \rtimes K$.

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which shows that $T_S(L) \simeq (X\flat L) \rtimes K$.
In particular, every T_S-algebra $\xi_L^S \colon T_S(L) \to L$ is determined by

$$
\begin{array}{ccccc}
X\flat L & \xrightarrow{\ \overline{\kappa}\ } & T_S(L) & \xleftarrow{\ \overline{i}\ } & K \\
& \searrow{\scriptstyle \xi_L^X} & \Big\downarrow{\scriptstyle \xi_L^S} & \swarrow{\scriptstyle \varphi} & \\
& & L. & &
\end{array}
$$

The decomposition of algebras

Proposition

Two morphisms $\xi_L^X \colon X \flat L \to L$ *and* $\varphi \colon K \to L$ *induce a morphism* $\xi_L^S \colon T_S(L) \to L$*, which is a* T_S*-algebra, if and only if*

1. ξ_L^X *is an* X*-action*
2. $\varphi^* \chi_L = k^* \xi_L^X \colon X \flat L \to L$
3. *the map* $[1, \varphi\rangle \colon L \rtimes K \to L$ *is* X*-equivariant.*

Ternary semidirect product

Definition (Carrasco, Cegarra)

An object X is a ternary semidirect product of its subobjects A, B, C if

- *X is the join of A, B and C in its lattice of subobjects;*
- *A and $A \vee B$ are normal in $\mathcal{C}$;*
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In the categories **Grp** and **Lie**, ternary semidirect products are determined by actions $\xi_B^C, \xi_A^C, \xi_A^B$, and some additional function $C \times B \to A$, satisfying certain conditions.

Proposition (Bourn)

Let

$$
\begin{array}{ccccc}
K & \xrightarrow{k} & X & \underset{s}{\overset{p}{\rightleftarrows}} & C \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} & & \| \\
L & \xrightarrow[l]{} & Y & \underset{t}{\overset{q}{\rightleftarrows}} & C
\end{array}
$$

be a morphism of split short exact sequences. Then the folllowing conditions are equivalent :

1. $l\varphi$ *is normal in* $\mathcal{C}$.
2. (φ, ψ) *is normal in* $\mathbf{Pt}_C(\mathcal{C})$.

In that case the cokernel of (φ, ψ) *in* $\mathbf{Pt}_C(\mathcal{C})$ *coincides with the cokernel of* $l\varphi$ *in* $\mathcal{C}$.

Thus a ternary semidirect product of A, B, C is equivalent to a short exact sequence

$$0 \longrightarrow A \xrightarrow{j_A} A \rtimes B \xrightarrow{p_B} B \longrightarrow 0$$

in $\mathbf{Act}_C(\mathcal{C})$.

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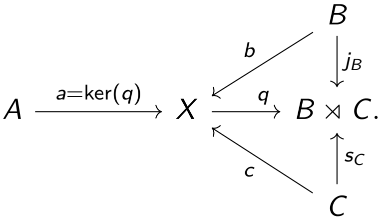
in $\mathbf{Act}_C(\mathcal{C})$. This yields a diagram

$$\begin{array}{ccccc}
A & \xrightarrow{\quad j_A \quad} & A \rtimes B & \underset{s_B}{\overset{p_B}{\rightleftarrows}} & B \\
\Vert & & \downarrow{\scriptstyle j_{A \rtimes B}} & & \downarrow{\scriptstyle j_B} \\
A & \xrightarrow{\quad a \quad} & (A \rtimes B) \rtimes C & \xrightarrow{p_B \rtimes C} & B \rtimes C \\
\downarrow & & {\scriptstyle s_C}\uparrow\downarrow{\scriptstyle p_C} & & {\scriptstyle s_C}\uparrow\downarrow{\scriptstyle p_C} \\
0 & \longrightarrow & C & = & C
\end{array}$$

Thus a ternary semidirect product is also equivalent to a morphism $q\colon X \to B \rtimes C$, whose kernel is A and such that B and C factor through q.



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$$\begin{array}{ccccc} & & & & B \\ & & & \overset{b}{\swarrow} & \downarrow j_B \\ A & \xrightarrow{a=\ker(q)} & X & \xrightarrow{q} & B \rtimes C. \\ & & & \underset{c}{\nwarrow} & \uparrow s_C \\ & & & & C \end{array}$$

In particular, this is equivalent to a pushforward

$$\begin{array}{ccccc} K[q_\xi] & \longrightarrow & B + C & \xrightarrow{q_\xi} & B \rtimes C \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle [b,c]} & & \| \\ A & \xrightarrow[a]{} & X & \xrightarrow[q]{} & B \rtimes C. \end{array}$$

Theorem (D.)

A ternary semidirect product of A, B, C is determined by

- *actions* $\xi_A^{C+B} \colon (C+B)\flat A \to A$ *and* $\xi_B^C \colon C\flat B \to B$
- *a morphism* $\varphi \colon K[\xi_B^C] \to A$

such that either

1. φ *is* $(C+B)$*-equivariant and* $(\varphi \rtimes (C+B))^*\chi = [\ker(q_\xi), 1\rangle^* \xi_A^{C+B}$
2. $A \rtimes K[\xi_B^C] \to A$ *is* $(C+B)$*-equivariant and* $\varphi^*\chi_A = \ker(q_\xi)^*\xi_A^{C+B}$.

Actions by semidirect products

If $X \simeq A \rtimes (B \rtimes C)$, we have in particular

$$X \underset{t}{\overset{q}{\rightleftarrows}} B \rtimes C \underset{s_C}{\overset{p_C}{\rightleftarrows}} C.$$

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In fact, this makes (q, t) a split epimorphism in $\mathbf{Pt}_C(\mathcal{C})$. Thus we have

$$\mathbf{Pt}_{B\rtimes C}(\mathcal{C}) \simeq \mathbf{Pt}_{(B\rtimes C, p_C, s_C)}(\mathbf{Pt}_C(\mathcal{C})),$$

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or equivalently

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Furthermore, X is then a ternary semidirect product, with

$$A \rtimes (B \rtimes C) \simeq (A \rtimes B) \rtimes C.$$

Actions by semidirect products

In the diagram

$$\begin{array}{ccccc} K[q_{B,C}] & \longrightarrow & B + C & \xrightarrow{q_\xi} & B \rtimes C \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle [b,c]} & & \| \\ A & \xrightarrow[a]{} & X & \longrightarrow & B \rtimes C \end{array}$$

the lower short exact sequence splits iff $\varphi = 0$.

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the lower short exact sequence splits iff $\varphi = 0$.
Thus a ternary semidirect product $(A \rtimes B) \rtimes C$ corresponds to a split extension in $\mathbf{Act}_C(\mathcal{C})$ (and is then isomorphic to $A \rtimes (B \rtimes C)$) if and only if $\varphi = 0$.

Comparing the results

Given an action ξ_L^X of X on L and $\varphi\colon K \to L$, there is a corresponding pushforward exact sequence if and only if

Version 1

1. φ is X-equivariant
2. $(\varphi \rtimes X)^*\chi = [k, 1\rangle^*\xi$.

Version 2

1. $\varphi^*\chi_L = k^*\xi_L^X$
2. $[1, \varphi\rangle\colon L \rtimes K \to L$ is X-equivariant.

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Version 2

1. $\varphi^*\chi_L = k^*\xi_L^X$
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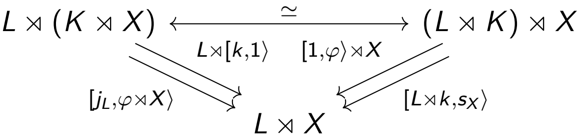
Both versions are equivalent to the existence of a pair of maps



Proposition

There exists a monad on $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ whose algebras give the structure of ternary semidirect products.

The functor part of the monad is defined by

$$(A, B, C) \mapsto (T_S(A), C\flat B, C)$$

where S denotes the short exact sequence

$$K[\mu^C_B] \longrightarrow C + C\flat B \xrightarrow{[i_C, \kappa_{C,B}]} C + B.$$