Regularity and complete distributivity in fuzzy metric spaces and formal contexts

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Motivation (I)

- The category **Sup** of complete sup-lattices and sup-preserving maps is $*$-autonomous, with tensor product classifying bimorphisms.
- Various constructions for this tensor product are known [Banaschewski & Nelson 1976, Joyal & Tierney 1984, Mowatt 1968, Shmuely 1974].
- The nuclear/dualisable objects in **Sup** are the completely distributive complete lattices [Higgs & Rowe 1984].

- Aim: obtain quantitative versions of these results (quantale-enriched).
- Some results are already known [Eklund et al 2018, Tholen 2024].
- Advantage: categorical constructions/proofs; choice-free; exhibit those results for **Sup** which depend on $\mathbb{2}$ being self-dual.

Quantales

- A (commutative) quantale $\mathcal{V}$ is a (commutative) monoid in **Sup**:

  $(\mathcal{V}, \vee, \perp)$ is a complete sup-lattice

  $(\mathcal{V}, \otimes, e)$ is a commutative monoid such that $- \otimes -$ preserves arbitrary sups

  Consequence: every $- \otimes v : \mathcal{V} \to \mathcal{V}$ has a right adjoint $[v, -] : \mathcal{V} \to \mathcal{V}$

- Typical examples
  - $(\mathbb{2}, \wedge, 1)$
  - $([0, \infty]^{op}, +, 0)$
  - $([0, 1], \otimes, 1)$, with $\otimes$ the usual product/min/Łukasiewicz product
  - The quantale of left continuous distribution functions $\Delta = \{f : [0, \infty] \to [0, 1] \mid f(a) = \bigvee_{b<a} f(b)\}$

$\mathcal{V}$-categories and $\mathcal{V}$-functors

- Let $\mathcal{V}$ be a commutative quantale. A (small) $\mathcal{V}$-enriched category $\mathcal{A}$ consists of a set of objects, together with a $\mathcal{V}$-valued relation (usually called $\mathcal{V}$-hom, or $\mathcal{V}$-distance, or $\mathcal{V}$-metric)

  $$\mathcal{A}(-,-) : \mathcal{A} \times \mathcal{A} \to \mathcal{V}$$

  satisfying

  $$e \leq \mathcal{A}(a,a) \quad \text{and} \quad \mathcal{A}(a'',a') \otimes \mathcal{A}(a',a) \leq \mathcal{A}(a'',a)$$

- A $\mathcal{V}$-enriched functor $f : \mathcal{A} \to \mathcal{B}$ consists of an assignment of objects $a \in \mathcal{A} \mapsto fa \in \mathcal{B}$ such that

  $$\mathcal{A}(a',a) \leq \mathcal{B}(fa',fa)$$

- **Remark** Each $\mathcal{V}$ category $\mathcal{A}$ carries an underlying order $a' \leq a \iff e \leq \mathcal{A}(a',a)$. The $\mathcal{V}$-category $\mathcal{A}$ is called separated if $\leq$ is antisymmetric. Each $\mathcal{V}$-functor is monotone wrt this underlying order.

$\mathcal{V}$-categories and $\mathcal{V}$-functors

Examples

- $\mathcal{V} = (\mathbb{2}, \wedge, 1)$: ordered sets & monotone maps [Lawvere 1973]
- $\mathcal{V} = ([0,\infty]^{op}, +, 0)$: (generalised) metric spaces & non-expansive maps [Lawvere 1973]
- $\mathcal{V} = \Delta$: probabilistic metric spaces (the Δ-hom $\mathcal{A}(a', a)$ evaluated at $p \in [0,1]$ can be interpreted as "probability that the distance from a' to a is less then p") & maps such that "the probability that "the distance from a' to a is less than p" is less than or equal to the "probability that the distance from fa' to fa is less than p"" [Menger 1942]

$\mathcal{V}$-categories and $\mathcal{V}$-functors

Denote as usual by $\mathcal{V}$-**Cat** the (2-)category of $\mathcal{V}$-categories and $\mathcal{V}$-functors.

Recall that $\mathcal{V}$-**Cat** is symmetric monoidal closed:

- The tensor product $\mathcal{A} \otimes \mathcal{B}$ of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ has as objects pairs (a, b) with $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $\mathcal{V}$-homs
$$(\mathcal{A} \otimes \mathcal{B})((a', b'), (a, b)) = \mathcal{A}(a', a) \otimes \mathcal{B}(b', b)$$

- The unit for the tensor product is the $\mathcal{V}$-category $\mathbb{1}$, with one object 0 and corresponding $\mathcal{V}$-hom given by $\mathbb{1}(0, 0) = e$.

- The internal hom between two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ is the $\mathcal{V}$-category of $\mathcal{V}$-functors $[\mathcal{A}, \mathcal{B}]$ from $\mathcal{A}$ to $\mathcal{B}$, with $\mathcal{V}$-distances
$$[\mathcal{A}, \mathcal{B}](f', f) = \bigwedge_{a} \mathcal{B}(f'a, fa)$$

$\mathcal{V}$-distributors

- A $\mathcal{V}$-distributor $\mathcal{A} \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!\!\mapstochar\quad\,} \mathcal{B}$ is a $\mathcal{V}$-functor $\varphi : \mathcal{B}^{op} \otimes \mathcal{A} \to \mathcal{V}$ (a "monotone" $\mathcal{V}$-valued relation)

  Particular cases: $\begin{cases} \mathbb{1} \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!\!\mapstochar\quad\,} \mathcal{A} & \text{contravariant presheaves} \\ & (\mathcal{V}\text{-valued "downsets"}) \\ \mathcal{A} \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!\!\mapstochar\quad\,} \mathbb{1} & \text{covariant presheaves} \\ & (\mathcal{V}\text{-valued "upsets"}) \end{cases}$

- $\mathcal{V}$-distributors compose by "matrix multiplication": the composite $\mathcal{A} \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!\!\mapstochar\quad\,} \mathcal{B} \overset{\psi}{\longrightarrow\!\!\!\!\!\!\!\!\!\mapstochar\quad\,} \mathcal{C}$ is

$$(\psi \otimes \varphi)(c, a) = \bigvee_{b} \psi(c, b) \otimes \varphi(b, a)$$

- The identity $\mathcal{V}$-distributor on a $\mathcal{V}$-category $\mathcal{A}$ is the $\mathcal{V}$-hom $\mathcal{A}(-,-)$

- Denote by **$\mathcal{V}$-Dist** the bicategory (quantaloid) of $\mathcal{V}$-enriched categories, $\mathcal{V}$-distributors and $\mathcal{V}$-natural transformations.

$\mathcal{V}$-distributors

- $\mathcal{V}$-**Dist** has all right extensions and liftings

$$\psi \otimes \varphi \leq \xi \quad \iff \quad \psi \leq \xi \swarrow \varphi \quad \iff \quad \varphi \leq \psi \searrow \xi$$

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left=20, "\psi\searrow\xi"{name=U}] \arrow[rr, bend right=20, "\varphi"'{name=D}] \arrow[dr, bend right=30, "\xi"'{name=X}] \arrow[dr, bend left=15, "\psi\otimes\varphi"{name=T}] & & \mathcal{B} \arrow[dl, bend right=15, "\psi"'{name=P}] \arrow[dl, bend left=30, "\xi\swarrow\varphi"{name=Q}] \\
& \mathcal{C} &
\arrow[from=D, to=U, dashed]
\arrow[from=T, to=X, dashed]
\arrow[from=P, to=Q, dashed]
\end{tikzcd}$$

where

$(\xi \swarrow \varphi)(c, b) = \bigwedge_a [\varphi(b, a), \xi(c, a)]$ and $(\psi \searrow \xi)(b, a) = \bigwedge_c [\psi(c, b), \xi(c, a)]$

- These induce the triple adjunction

$\mathcal{V}\text{-}\mathbf{Dist}(\mathcal{A}, \mathcal{B})(\varphi, \psi \searrow \xi) \cong \mathcal{V}\text{-}\mathbf{Dist}(\mathcal{A}, \mathcal{C})(\psi \otimes \varphi, \xi) \cong \mathcal{V}\text{-}\mathbf{Dist}(\mathcal{B}, \mathcal{C})(\psi, \xi \swarrow \varphi)$

Cocomplete $\mathcal{V}$-categories

Denote now by **$\mathcal{V}$-Sup** the (2-)category of separated cocomplete $\mathcal{V}$-categories and cocontinuous $\mathcal{V}$-functors.

There are many equivalent descriptions of (separated) cocomplete $\mathcal{V}$-categories (and correspondingly, of cocontinuous $\mathcal{V}$-functors):

- (Separated) $\mathcal{V}$-categories having all (small) colimits
- (Strict) algebras for the free cocompletion 2-monad $\mathbb{D}$ on **$\mathcal{V}$-Cat**
- Injective $\mathcal{V}$-categories (wrt fully faithful $\mathcal{V}$-functors) [Hofmann 2006]
- Complete sup-lattices endowed with an action of the quantale $\mathcal{V}$ (hence the notation **$\mathcal{V}$-Sup**) [Joyal & Tierney 1984]
- Algebras for the $\mathcal{V}$-valued powerset monad on **Set** [Pedicchio & Tholen 1989]

Free cocompletion 2-monad on $\mathcal{V}$-**Cat**

- Let $\mathbb{D}\mathcal{A}$ be $[\mathcal{A}^{op}, \mathcal{V}]$, the $\mathcal{V}$-category of contravariant presheaves on $\mathcal{A}$ ($\mathcal{V}$-valued "downsets").

- The correspondence $\mathcal{A} \mapsto \mathbb{D}\mathcal{A}$ produces a monad $\mathbb{D} : \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$ having as unit the Yoneda embedding

$$y_{\mathcal{A}} : \mathcal{A} \to \mathbb{D}\mathcal{A} \ , \ \ y(a) = \mathcal{A}(-, a)$$

  and multiplication the $\mathcal{V}$-"union" of downsets.

- The action of $\mathbb{D}$ on a $\mathcal{V}$-functor $f : \mathcal{A} \to \mathcal{B}$ is

$$\mathbb{D}f = Lan_{y_{\mathcal{A}}}(y_{\mathcal{B}} \circ f) : \mathbb{D}\mathcal{A} \to \mathbb{D}\mathcal{B}$$

  Recall that each $\mathbb{D}f$ has a right adjoint which itself has a right adjoint $\mathbb{D}_\forall$:

$$\begin{tikzcd}
\mathbb{D}\mathcal{A} \arrow[r, shift left=5, "\mathbb{D}f"] \arrow[r, phantom, shift left=2.5, "\perp"] & \mathbb{D}\mathcal{B} \arrow[l] \arrow[l, phantom, shift left=2.5, "\perp"] \\
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}\mathcal{A} \arrow[r, shift right=5, "\mathbb{D}_\forall f"'] & \mathbb{D}\mathcal{B}
\end{tikzcd}$$

Free cocompletion 2-monad on $\mathcal{V}$-**Cat**

- $\mathbb{D}$ is the free cocompletion monad on $\mathcal{V}$-**Cat** [Kelly 1982, Stubbe 2006]; as such, it is a Kock-Zöberlein-monad.

- A (pseudo) $\mathbb{D}$-algebra is a cocomplete $\mathcal{V}$-category $\mathcal{A}$, with structure provided by the left adjoint $sup_{\mathcal{A}}$ of $y_{\mathcal{A}}$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift right, "y_{\mathcal{A}}"'] \arrow[r, phantom, "\perp"] & \mathbb{D}\mathcal{A} \arrow[l, shift right, "sup_{\mathcal{A}}"']
\end{tikzcd}$$

- A (pseudo) $\mathbb{D}$-homomorphism $f : (\mathcal{A}, sup_{\mathcal{A}}) \to (\mathcal{B}, sup_{\mathcal{B}})$ is a cocontinuous $\mathcal{V}$-functor.

$$\begin{tikzcd}
\mathbb{D}\mathcal{A} \arrow[r, "\mathbb{D}f"] \arrow[d, "sup_{\mathcal{A}}"'] & \mathbb{D}\mathcal{B} \arrow[d, "sup_{\mathcal{B}}"] \\
\mathcal{A} \arrow[r, "f"] & \mathcal{B}
\end{tikzcd}$$

Tensor product of cocomplete $\mathcal{V}$-categories

- Recall $\mathcal{V}$-**Sup**, the category of separated cocomplete $\mathcal{V}$-categories and cocontinuous $\mathcal{V}$-functors (the category of (strict) $\mathbb{D}$-algebras).
- Being the free cocompletion monad, $\mathbb{D}$ is commutative [López Franco 2011], therefore $\mathcal{V}$-**Sup** is symmetric monoidal closed:
    - The tensor product $\otimes_{\mathcal{V}\text{-}\mathbf{Sup}}$ classifies bimorphisms [Banaschewski & Nelson 1976] ($\mathcal{V} = \mathbb{2}$), [Joyal & Tierney 1984]

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{B} \arrow[r, "universal", "bimorphism"'] \arrow[dr, "bimorphism"'] & \mathcal{A} \otimes_{\mathcal{V}\text{-}\mathbf{Sup}} \mathcal{B} \arrow[d, dotted, "morphism"] \\
 & \mathcal{C}
\end{tikzcd}$$

    - The unit is $\mathbb{D}\mathbb{1} = \mathcal{V}$
    - The internal hom is $\mathcal{V}\text{-}\mathbf{Sup}(\mathcal{A}, \mathcal{B})$

Tensor product of cocomplete $\mathcal{V}$-categories

- **Theorem** [B 2024]
  The tensor product $\mathcal{A} \otimes_{\mathcal{V}\text{-}\mathbf{Sup}} \mathcal{B}$ of two cocomplete $\mathcal{V}$-categories $\mathcal{A}, \mathcal{B}$ can be obtained as the following inverter below:

$$\begin{tikzcd}
\mathcal{A} \otimes_{\mathcal{V}\text{-}\mathbf{Sup}} \mathcal{B} \arrow[r, hook, "j"] & \mathbb{D}(\mathcal{A} \otimes \mathcal{B}) \arrow[r, shift left=1.5, "\mathbb{D}(y_{\mathcal{A}} \otimes y_{\mathcal{B}})"] \arrow[r, shift right=1.5, "\mathbb{D}_{\forall}(y_{\mathcal{A}} \otimes y_{\mathcal{B}})"'] \arrow[r, phantom, "\Downarrow"] & \mathbb{D}(\mathbb{D}\mathcal{A} \otimes \mathbb{D}\mathcal{B})
\end{tikzcd}$$

  In particular, $\mathcal{A} \otimes_{\mathcal{V}\text{-}\mathbf{Sup}} \mathcal{B}$ is reflective in $\mathbb{D}(\mathcal{A} \otimes \mathcal{B})$.

- Proof sketch. The tensor product in the category of algebras for a monad is usually computed as a (reflexive) coequalizer. The monad $\mathbb{D}$ being KZ, the coequalizer turns into a coinverter. Applying the $3x3$ lemma for (reflexive) coinverters and the duality between complete and cocomplete $\mathcal{V}$-categories leads to the result.

Tensor product of cocomplete $\mathcal{V}$-categories

Remarks

- This description of $\otimes_{\mathcal{V}\text{-}\mathbf{Sup}}$ generalises the one for the tensor product of sup-lattices by G-ideals (down-sets of the cartesian product join-closed in either coordinate) [Shmuely 1974]
- The universal bimorphism

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{y} \mathbb{D}(\mathcal{A} \otimes \mathcal{B}) \xrightarrow{q} \mathcal{A} \otimes_{\mathcal{V}\text{-}\mathbf{Sup}} \mathcal{B}$$

  is dense and point-separating with respect to the forgetful functor $\mathcal{V}\text{-}\mathbf{Sup} \to \mathcal{V}\text{-}\mathbf{Cat}$ (where q is the reflector).

  In particular, every object of $\mathcal{A} \otimes_{\mathcal{V}\text{-}\mathbf{Sup}} \mathcal{B}$ can be canonically represented as a colimit of "elementary tensors" $a \otimes_{\mathcal{V}\text{-}\mathbf{Sup}} b = q \circ y(a, b)$.

Corollary

- The monoidal structure of $\mathcal{V}$-**Sup** restricts to the full subcategory $\mathcal{V}\text{-}\mathbf{CCD}_{sup}$ (see p. 18-19)

More on the tensor product of cocomplete $\mathcal{V}$-categories

- There is a duality $\mathcal{V}\text{-}\mathbf{Sup} \cong \mathcal{V}\text{-}\mathbf{Sup}^{op}$, sending $\mathcal{A}$ to $\mathcal{A}^{op}$ and $f : \mathcal{A} \to \mathcal{B}$ to $g^{op} : \mathcal{B}^{op} \to \mathcal{A}^{op}$, where $f \dashv g$
- In particular, $\mathcal{A}^{op} \cong \mathcal{V}\text{-}\mathbf{Sup}(\mathcal{V}, \mathcal{A}^{op}) \cong \mathcal{V}\text{-}\mathbf{Sup}(\mathcal{A}, \mathcal{V}^{op})$
- This implies that $\mathcal{V}\text{-}\mathbf{Sup}$ is not only symmetric monoidal closed, but also $*$-autonomous, with dualiser $\mathcal{V}^{op}$
- Consequently, the tensor product can be equivalently described using Galois connections [Eklund et al 2018, Tholen 2024]

$$\mathcal{A} \otimes_{\mathcal{V}\text{-}\mathbf{Sup}} \mathcal{B} \cong \mathcal{V}\text{-}\mathbf{Sup}(\mathcal{A}, \mathcal{B}^{op})^{op}$$

$\mathcal{V}$-"Sup is good food" (R. Blute, FMCS 2022)

- What about other (monoidal) features of $\mathcal{V}\text{-}\mathbf{Sup}$?

Nuclearity/dualisability

- Grothendieck introduced in Functional Analysis the concept of nuclearity for objects and morphisms, in order to mimic finite dimensionality behaviour (for objects) and matrix calculus (for arrows) [Grothendieck 1955].

- Nuclearity (nowadays called dualisability in Category Theory) can be defined in the more general context of (symmetric) monoidal closed categories [Kelly 1972, Saavedro-Rivano 1972, Kelly & Laplaza 1980, Rowe 1988].

Dualisability

- In a symmetric monoidal closed category, an arrow $f : \mathcal{A} \to \mathcal{B}$ is nuclear/dualisable if the associated $\mathbb{1} \to [\mathcal{A}, \mathcal{B}]$ factorises through the canonical arrow $\mathcal{B} \otimes [\mathcal{A}, \mathbb{1}] \to [\mathcal{A}, \mathcal{B}]$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, dotted] \arrow[rr, bend right] & \mathcal{B} \otimes [\mathcal{A}, \mathbb{1}] \arrow[r] & {[\mathcal{A}, \mathcal{B}]}
\end{tikzcd}$$

- An object $\mathcal{A}$ is nuclear/dualisable if any of the following equivalent conditions hold:
  - $\mathcal{B} \otimes [\mathcal{A}, \mathcal{C}] \cong [\mathcal{A}, \mathcal{B} \otimes \mathcal{C}]$ for all $\mathcal{B}, \mathcal{C}$
  - $\mathcal{A} \otimes [\mathcal{A}, \mathbb{1}] \cong [\mathcal{A}, \mathcal{A}]$ (that is, $id_{\mathcal{A}}$ is nuclear)

  [Higgs & Rowe 1989, Kelly 1972, Kelly & Laplaza 1980]

  If this is the case, then $\mathcal{A}^* = [\mathcal{A}, \mathbb{1}]$ is the dual of $\mathcal{A}$ and there are arrows $\mathbb{1} \to \mathcal{A} \otimes \mathcal{A}^*$, $\mathcal{A}^* \otimes \mathcal{A} \to \mathbb{1}$ satisfying the usual triangular identities.

- If all objects are nuclear/dualisable, the category is compact closed.

Dualisability in $\mathcal{V}$-**Sup**

- A cocomplete $\mathcal{V}$-category $\mathcal{A}$ is completely distributive ($\mathcal{V}$-**CCD**) if the left adjoint to the Yoneda embedding (the $\mathcal{V}$-functor taking "suprema") has itself a left adjoint ($\mathcal{V}$-valued analogue of the totally below relation):

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left=6, "\Downarrow_{\mathcal{A}}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "y_{\mathcal{A}}"'] & \mathbb{D}\mathcal{A} \arrow[l, "sup_{\mathcal{A}}" description] \end{tikzcd}$$

[Stubbe 2007]

- More on $\mathcal{V}$-**CCD**s:
  - Projective objects of $\mathcal{V}$-**Sup** [Stubbe 2007]
  - Algebras for the double dualisation monad $[[-,\mathcal{V}],\mathcal{V}]$ on $\mathcal{V}$-**Cat** [Băbuș & Kurz 2016, Stubbe 2017]
  - (Separated) $\mathcal{V}$-**CCD**s and continuous and cocontinuous $\mathcal{V}$-functors form an (infinitary) variety, thus have an equational presentation [B & Kurz 2021]

Dualisability in $\mathcal{V}$-**Sup**

- **Lemma** [B 2024] A free cocomplete $\mathcal{V}$-category $\mathbb{D}\mathcal{A}$ is dualisable, with dual $\mathbb{D}(\mathcal{A}^{op})$.

- **Theorem** [B 2024] The dualisable objects in $\mathcal{V}$-**Sup** are precisely the $\mathcal{V}$-**CCD**s.

- **Corollary** The full subcategory of $\mathcal{V}$-**Sup** consisting of completely distributive complete $\mathcal{V}$-categories and cocontinuous $\mathcal{V}$-functors $\mathcal{V}$-$\mathbf{CCD}_{sup}$ is compact closed with respect to the tensor product and internal hom inherited from $\mathcal{V}$-**Sup**, the dual of a $\mathcal{V}$-**CCD** $\mathcal{A}$ being $\mathcal{A}^{op}$.

- **Question** What are the nuclear arrows in $\mathcal{V}$-**Sup**? (notice that in **Sup**, these are Raney's tight maps)

Dualisability in $\mathcal{V}$-**Sup**

Examples

- The free cocompletion $\mathbb{D}\mathcal{A}$ of any $\mathcal{V}$-category $\mathcal{A}$ is $\mathcal{V}$-**CCD** [Lai & Zhang 2006, Stubbe 2007]
- In particular, the quantale $\mathcal{V}$ is itself $\mathcal{V}$-**CCD** as a $\mathcal{V}$-category,
- Retracts of free cocompletions are $\mathcal{V}$-**CCD** (in fact, all $\mathcal{V}$-**CCD**s arise in that way) [Stubbe 2007]
- More (explicit) examples?

Motivation (II)

- **Theorem** [Raney 1960, Bandelt 1980]

  For a partially ordered set $(A, \leq)$, the following are equivalent:

  - The Dedekind-MacNeille completion of $(A, \leq)$ is a completely distributive complete lattice
  - $\not\leq$ is a regular relation on A

- **Questions**
  - Can the above result be generalised from ordered sets to quantale-enriched categories?

    Yes (with extra assumptions on the quantale $\mathcal{V}$)

  - Are there interesting applications?

    Yes (e.g. quantale-valued formal concept analysis)

Regularity

- An element x of a semigroup S is called regular if there is some $y \in S$ satisfying $xyx = x$ ("generalised inverse" for x) [Moore 1920, von Neumann 1936, Green 1951].

  An arrow $f : A \to B$ in a category $\mathcal{A}$ is called regular if there is $g : B \to A$ with $f \circ g \circ f = f$ [MacLane 1971].

- **Examples**
  - The apartness relation $\in$ between a set A and its powerset $\mathcal{P}(A)$ is regular.
  - Idempotent relations, in particular orders or equivalence relations, are regular.
  - Any real or complex matrix M is regular ($MM^{+}M = M$, where M^{+} is the Moore-Penrose inverse of M)

Relations, Galois connections and regularity

- A relation between sets $\varphi : A \nrightarrow B$ induces a covariant adjunction (axiality)

  $$\begin{tikzcd} \mathcal{P}(A) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{P}(B) \arrow[l, shift left] \end{tikzcd} \quad , \quad \begin{cases} X \subseteq A \mapsto \{b \in B \mid \exists a \in X \,.\, \varphi(a,b)\} \\ Y \subseteq B \mapsto \{a \in A \mid \varphi(a,b) \Rightarrow b \in Y\} \end{cases}$$

  and a contravariant adjunction (polarity)

  $$\begin{tikzcd} \mathcal{P}(A) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{P}(B)^{op} \arrow[l, shift left] \end{tikzcd} \quad , \quad \begin{cases} X \subseteq A \mapsto \{b \in B \mid \forall a \,.\, a \in X \Rightarrow \varphi(a,b)\} \\ Y \subseteq B \mapsto \{a \in A \mid \forall b \,.\, b \in Y \Rightarrow \varphi(a,b)\} \end{cases}$$

  between powersets.
- Conversely, each covariant or contravariant adjunction between powersets arises from a binary relation as above [Birkhoff 1940, Everett 1944, MacNeille 1937, Ore 1944]
- A relation $\varphi : A \nrightarrow B$ is regular iff the complete lattice of fixed points of the induced adjunction between powersets $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is completely distributive [Raney 1953], [Zaretskiĭ 1962], [Xu & Liu 2004].

Regularity in $\mathcal{V}$-**Dist**

- **Regularity in $\mathcal{V}$-Cat**

  A $\mathcal{V}$-distributor $\mathcal{A} \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!\!\mid\;\;\;\;} \mathcal{B}$ is regular if there exists $\mathcal{B} \overset{\psi}{\longrightarrow\!\!\!\!\!\!\!\!\!\mid\;\;\;\;} \mathcal{A}$ such that $\varphi = \varphi \otimes \psi \otimes \varphi$.

  Equivalently, φ is regular if $\varphi = \varphi \otimes (\varphi \searrow \varphi \swarrow \varphi) \otimes \varphi$.

  Idempotent $\mathcal{V}$-distributors, or (left/right) adjoint $\mathcal{V}$-distributors are regular.

- Similar characterisation of regular distributors in terms of fixed points of induced adjunctions? Yes, see next slides.

$\mathcal{V}$-distributors and adjunctions

- Recall that a $\mathcal{V}$-distributor $\mathcal{A} \overset{\varphi}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathcal{B}$ is a $\mathcal{V}$-functor $\varphi : \mathcal{B}^{op} \otimes \mathcal{A} \to \mathcal{V}$ (a "monotone" $\mathcal{V}$-valued relation)
- Every $\mathcal{V}$-distributor $\mathcal{A} \overset{\varphi}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathcal{B}$ induces adjunctions between covariant/contravariant presheaves:
  - The Kan adjunctions[1]

$$\begin{tikzcd}[\mathcal{A}^{op}, \mathcal{V}] \arrow[r, shift left=2, "\varphi \otimes -"] \arrow[r, phantom, "\perp"] & {[\mathcal{B}^{op}, \mathcal{V}]} \arrow[l, shift left=2, "\varphi \searrow -"]\end{tikzcd} \qquad \begin{tikzcd}[\mathcal{A}, \mathcal{V}]^{op} \arrow[r, shift left=2, "- \otimes \varphi"] \arrow[r, phantom, "\perp"] & {[\mathcal{B}, \mathcal{V}]^{op}} \arrow[l, shift left=2, "- \swarrow \varphi"]\end{tikzcd}$$

  - The Isbell adjunction

$$\begin{tikzcd}[\mathcal{B}^{op}, \mathcal{V}] \arrow[r, shift left=2, "- \searrow \varphi"] \arrow[r, phantom, "\perp"] & {[\mathcal{A}, \mathcal{V}]^{op}} \arrow[l, shift left=2, "\varphi \swarrow -"]\end{tikzcd}$$

- These adjunctions go back to [Bénabou 1973, Lambek 1966] in category theory.

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[1] The terminology is borrowed from [Shen & Zhang 2013].

$\mathcal{V}$-distributors and adjunctions

- For the Kan adjunction associated to a $\mathcal{V}$-distributor $\mathcal{A} \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathcal{B}$, let **FixKan**($\varphi$) denote the cocomplete $\mathcal{V}$-category of fixed points of the adjunction:

$$\begin{tikzcd}
{[\mathcal{A}^{op}, \mathcal{V}]} \arrow[r, shift left=1.5ex, "\varphi\otimes -"] \arrow[r, phantom, "\perp"] & {[\mathcal{B}^{op}, \mathcal{V}]} \arrow[l, shift left=1.5ex, "\varphi\searrow -"]
\end{tikzcd}$$

- **Theorem** [Lai & Shen 2018]

  FixKan(φ) is $\mathcal{V}$-**CCD**^{op} $\Rightarrow$ φ is regular $\Rightarrow$ **FixKan**(φ) is $\mathcal{V}$-**CCD**

  The converse implications hold for every $\mathcal{V}$-distributor φ iff $\mathcal{V}$ is a Girard quantale.

- Being $\mathcal{V}$-**CCD** and $\mathcal{V}$-**CCD**^{op} are equivalent notions when $\mathcal{V} = \mathbb{2}$, but not in general!

- Interpret the above result as an instance of the microcosm principle $\mathcal{V}$-**Sup** is $*$-autonomous, and for things to go as expected, $\mathcal{V}$ must also be so (i.e. Girard!)

$\mathcal{V}$-distributors and adjunctions

- For the Isbell adjunction associated to a $\mathcal{V}$-distributor $\mathcal{A} \overset{\varphi}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathcal{B}$, let **FixIsbell**$(\varphi)$ denote the cocomplete $\mathcal{V}$-category of fixed points of the adjunction[2]:

$$\begin{tikzcd} {[\mathcal{B}^{op}, \mathcal{V}]} \arrow[r, shift left, "- \searrow \varphi"] \arrow[r, phantom, "\perp"] & {[\mathcal{A}, \mathcal{V}]^{op}} \arrow[l, shift left, "\varphi \swarrow -"] \end{tikzcd}$$

- Formal Concept Analysis: **FixIsbell**(φ) is the concept lattice/$\mathcal{V}$-category associated to the context $(\mathcal{A}, \mathcal{B}, \varphi)$

- For $\varphi = \mathcal{A}(-,-)$ is the $\mathcal{V}$-valued hom of a $\mathcal{V}$-category $\mathcal{A}$, **FixIsbell**(φ) is the Dedekind-MacNeille-Isbell completion of $\mathcal{A}$.

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[2] Also known as the nucleus of φ [Pavlović & Hughes 2020].

Connecting $\mathcal{V}$-**CCD** and regularity of $\mathcal{V}$-distributors

Proposition [B 2025]

If $\mathcal{V}$ is a Girard quantale with linear negation denoted $(-)^{\perp}$, then the Isbell completion

$$\mathbf{FixIsbell}(\mathcal{A})$$

of a $\mathcal{V}$-category $\mathcal{A}$ is $\mathcal{V}$-**CCD** iff

$$\mathcal{A}^{\perp}(-,-) = (-)^{\perp} \circ \mathcal{A}(-,-)$$

is a regular $\mathcal{V}$-distributor.

Connecting $\mathcal{V}$-**CCD** and regularity of $\mathcal{V}$-distributors

Proof sketch

- For each $\mathcal{V}$-category $\mathcal{A}$, the following diagram commutes [Willerton 2021]:

$$\begin{tikzcd}
{[\mathcal{A}^{op}, \mathcal{V}]} \arrow[r, shift left, "\mathcal{A}^\perp \otimes -"] \arrow[r, phantom, "\perp"] \arrow[d, equal] & {[\mathcal{A}^{op}, \mathcal{V}]} \arrow[l, shift left, "\mathcal{A}^\perp \searrow -"] \arrow[d, "(-)^\perp"] \\
{[\mathcal{A}^{op}, \mathcal{V}]} \arrow[r, shift left, "- \searrow \mathcal{A}"] \arrow[r, phantom, "\perp"] & {[\mathcal{A}, \mathcal{V}]^{op}} \arrow[l, shift left, "\mathcal{A} \swarrow -"]
\end{tikzcd}$$

- The $\mathcal{V}$-category of fixed points for the lower adjunction is **FixIsbell**$(\mathcal{A})$.
- The $\mathcal{V}$-category of fixed points of the upper adjunction **FixKan**$(\mathcal{A}^\perp)$ is $\mathcal{V}$-**CCD** iff $\mathcal{A}^\perp(-,-)$ is a regular $\mathcal{V}$-distributor.

References

- [B 2025]
  On the tensor product of quantale-enriched completely distributive categories
  `arXiv2501.12014`

- [B 2025] (in preparation)
  Isbell completion and regularity for quantale-enriched categories

Thank you for your attention!