From Fuzzy Markov Categories towards Imprecise Probability

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We investigate three different varieties of uncertainty:

- classical, discrete probability distributions $\mathrm{D}X$ over a set X
- imprecise probability, by which we mean certain subsets $S \subseteq \mathrm{D}X$
- fuzzy sets, i. e. , functions $X \to \mathcal{V}$ for quantales $\mathcal{V}$ (e. g, $\mathcal{V} = [0, 1]$)

Fuzzy sets have a long tradition as mathematical models for imprecision and vagueness[Zad65], with fuzzy powerset monads $F_{\mathcal{V}}$ as categorical formalisation [Man76].
More recently, CD categories [CJ19] and Markov categories [Fri20] have proven useful as categorical framework for various versions of probabilistic reasoning: they axiomatise categories like the category of Markov kernels, i.e. the Kleisli category Kl_{D} of the (discrete) distribution monad D.
We show that these two approaches to uncertainty are compatible, in the sense that Kleisli categories of $F_{\mathcal{V}}$ are CD categories or Markov categories (depending on the considered morphisms). They are also very similar, in that the underlying functors factorise over the contravariant powerset functor.
However, the two approaches are *in*compatible in that $\mathsf{Kl}_{F_{\mathcal{V}}}$ does *not* extend Kl_{D}. This would be desirable for a categorical notion of imprecise probability in the spirit of upper and lower probabilities $(a_x, b_x) \in [0, 1]^{2X}$ [Bor46].
Hence, we remain with an open question: is there a categorical description of imprecise probability?

References

[Bor46] Émile Borel. “Les probabilités et la vie”. In: (1946).

[CJ19] Kenta Cho and Bart Jacobs. “Disintegration and Bayesian inversion via string diagrams”. In: *Mathematical Structures in Computer Science* 29.7 (2019), pp. 938–971.

[Dem67] A. P. Dempster. "Upper and Lower Probabilities Induced by a Multivalued Mapping". In: *The Annals of Mathematical Statistics* 38.2 (1967), pp. 325–339. DOI: 10.1214/aoms/1177698950. URL: https://doi.org/10.1214/aoms/1177698950.

[Fri20] Tobias Fritz. "A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics". In: *Advances in Mathematics* 370 (2020), p. 107239.

[Man76] Ernest G Manes. "Algebraic Theories". In: *Graduate Texts in Mathematics* (1976).

[Sha76] G Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[Zad65] Lotfi A Zadeh. "Fuzzy sets". In: *Information and control* 8.3 (1965), pp. 338–353.