Pushforward and ternary semidirect products in semi-abelian categories

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Abstract.

Given a short exact sequence $K \xrightarrow{k} X \xrightarrow{q} Q$ in a semi-abelian category $\mathcal{C}$, and a morphism $\varphi\colon K \to L$, one can show [1] that φ can be extended to a morphism

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{q} & Q \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ L & \xrightarrow[l]{} & Y & \xrightarrow[q']{} & Q \end{array} \tag{1}$$

of short exact sequences if and only if there exists an action $\xi_L^X\colon X\flat L \to L$ of X on L such that φ is a morphism of X-actions $(K, \chi_K^X) \to (L, \xi_K^X)$ and

$$(\varphi \rtimes X)^* \chi_L^{L\rtimes X} = [k, 1\rangle^* \xi_L^X. \tag{2}$$

In this talk, we give a new interpretation of this result and explore some applications. To this end, we consider the co-slice categories of the category $\mathbf{SES}(C)$ of short exact sequences in $\mathcal{C}$. We exhibit an adjunction between each such co-slice and $\mathcal{C}$, and we show that pushforwards correspond to certain algebras for the induced monads.

As particular cases of monads induced by an adjunction of this type, we obtain the monads of internal actions $L \mapsto X\flat L$, as well as the monads of the form $L \mapsto K + L$ for a fixed object K. Furthermore, we show that the functor part of such a monad admits a natural decomposition as a semidirect product. Using this fact, we show that its algebras can similarly be decomposed into an action $\xi_L^X\colon X\flat L \to L$ and a morphism $\varphi\colon K \to L$. We then characterize the pairs (ξ_L^X, φ) that give rise to an algebra.

We will then explain higher order semidirect products (as defined by Carrasco and Cegarra [2, 3]) in semi-abelian categories can be constructed as pushforwards of this type, and how the monads of pushforwards can be used to describe the structure of these semidirect products.

References

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